Principal Congruence Links for Discriminant $D = -3$

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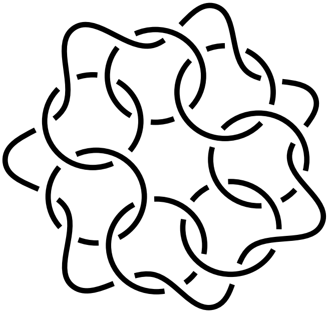
Overview

- Thurston congruence link, geometric description
- Bianchi orbifolds, congruence and principal congruence manifolds
- Results implying there are finitely many principal congruence links
- Overview for the case of discriminant $D = -3$
- Preliminaries for the construction
- Construction of two more examples
- Open questions

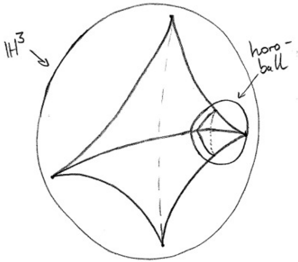
Thurston congruence link

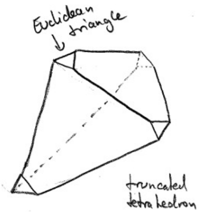


$M_{2+\zeta}^{-3}$

- Complement is non-compact finite-volume hyperbolic 3-manifold.
- Tesselated by 28 regular ideal hyperbolic tetrahedra.
- Tesselation is "regular", i.e., symmetry group takes every tetrahedron to every other tetrahedron in all possible 12 orientations.

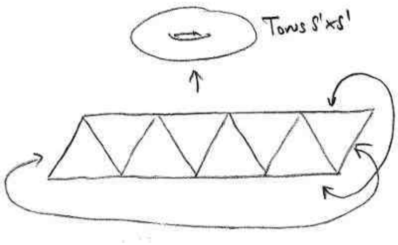
Cusped hyperbolic 3-manifolds





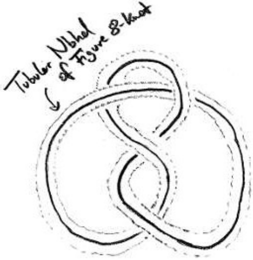
- Ideal hyperbolic tetrahedron does not include the vertices.
- Remove a small hororball. Ideal tetrahedron is topologically a truncated tetrahedron.
- Cut is a triangle with a Euclidean structure from hororsphere.

Cusped hyperbolic 3-manifolds have toroidal ends



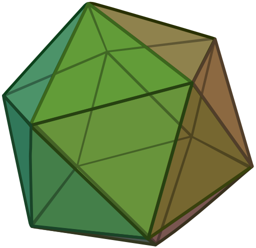
- Truncated tetrahedra form interior of a 3-manifold $\bar{M}$ with boundary.
- $\partial\bar{M}$ triangulated by the Euclidean triangles.
- $\partial\bar{M}$ is a torus.
- Ends (cusps) of hyperbolic manifold modeled on torus $\times$ interval.

Knot complements can be cusped hyperbolic 3-manifolds



- Cusp homeomorphic to a tubular neighborhood of a knot/link component.
- Figure-8 knot complement tesselated by two regular ideal tetrahedra.
- Hyperbolic metric near knot so dense that light never reaches knot.
- Complement still has finite volume.

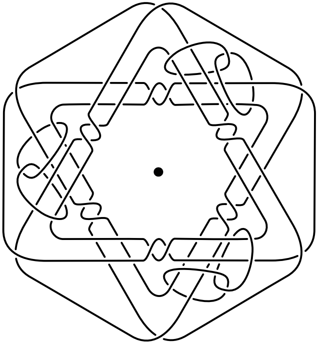
“Regular tesselations”



(Source: wikipedia)

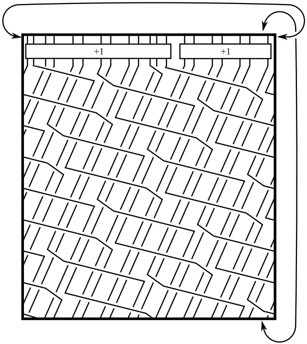
- Spherical 2-dimensional version of “regular tesselations”: Platonic solids.
- Person in a tile cannot tell through intrinsic measurements in what tile he or she is or at what edge he or she is looking at.

Two more examples



M_3^{-3}

54 regular ideal tetrahedra



$M_{2+2\zeta}^{-3}$

120 regular ideal tetrahedra

Thurston congruence link and the Klein quartic

$xy^3 + yz^3 + zx^3 = 0$

- Faces of ideal tetrahedra form immersed hyperbolic surface.
- Filling the punctures yields an algebraic curve in $\mathbb{C}P^2$: Klein quartic.
- Orientation-preserving symmetry group of the hyperbolic surface: $\mathrm{PSL}(2,7)$, the unique finite simple group of order 168.
- Thurston/Agol, "Thurston congruence link"

Bianchi orbifolds

- $\mathcal{O}_D$: ring of integers in $\mathbb{Q}(\sqrt{D})$. $D < 0, D \equiv 0, 1(4)$ discriminant.
- **Bianchi group**:

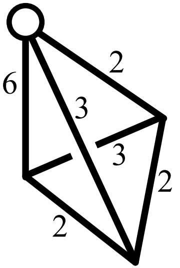
$$\mathrm{PGL}(2, \mathcal{O}_D) \quad \text{respectively} \quad \mathrm{PSL}(2, \mathcal{O}_D)$$

  is a discrete subgroup of $\mathrm{PGL}(2, \mathbb{C}) \cong \mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$.
- **Bianchi orbifold**:

$$\mathrm{M}_1^D = \frac{\mathbb{H}^3}{\mathrm{PGL}(2, \mathcal{O}_D)} \quad \text{respectively} \quad \frac{\mathbb{H}^3}{\mathrm{PSL}(2, \mathcal{O}_D)}.$$

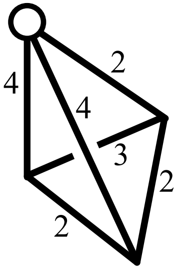
- Every cusped arithmetic hyperbolic manifold is commensurable with a Bianchi orbifold.

Bianchi orbifolds



M_1^{-3}

regular ideal tetrahedron
divided by
orientation-preserving
symmetries



M_1^{-4}

regular ideal octahedron
divided by
orientation-preserving
symmetries

Congruence subgroups

- Fix ideal I in $\mathcal{O}_D$.
- $\mathcal{O}_D \to \mathcal{O}_D/I$ induces map

$$p : \mathrm{PGL}(2, \mathcal{O}_D) \to \mathrm{PGL}(2, \mathcal{O}_D/I)$$

- **Congruence subgroup**:

$$p^{-1}(G) \text{ for some subgroup } G \subset \mathrm{PGL}(2, \mathcal{O}_D/I).$$

- **Principal congruence subgroup**:

$$\ker(p) = p^{-1}(0).$$

- **(Principal) congruence manifold/orbifold**: quotient of $\mathbb{H}^3$

$$\mathrm{M}_z^D = \frac{\mathbb{H}^3}{\ker\left(\mathrm{PGL}\left(2, \mathcal{O}_D\right) \to \mathrm{PGL}\left(2, \frac{\mathcal{O}_D}{\langle z \rangle}\right)\right)}$$

- Thurston congruence link complement is $\mathrm{M}_{2+\zeta}^{-3}$.

Cuspidal Cohomology, Baker's links

- Cuspidal cohomology yields an obstruction:
  If M_1^D can be covered by a link complement, then $D \in \mathcal{L}$ where $\mathcal{L} =$
  $$\{-3, -4, -7, -8, -11, -15, -19, -20, -23, -24, -31, -39, -47, -71\}.$$
- Mark Baker constructed "some" cover for each $D \in \mathcal{L}$, making it 'iff'.
- His covers are neither canonical nor regular.

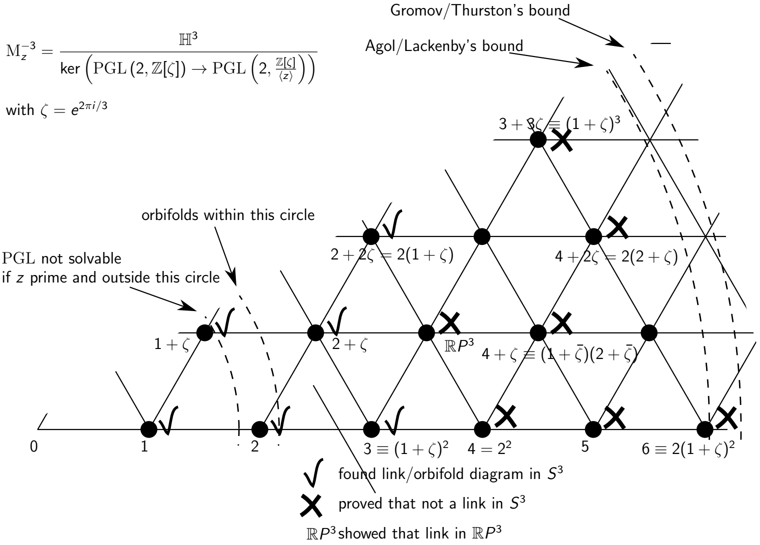
Finitely many principal congruence links

- **Gromov and Thurston 2π-Theorem:** Dehn filling cusps of a hyperbolic manifold along peripheral curves with length $> 2\pi$ yields hyperbolic manifold again.
  (Length measured on embedded hororballs)
- **Agol and Lackenby:** improved bound to > 6.
- **Corollary:** If the shortest curve on every cusp has length > 6, the manifold is not a link complement.
- Hence, only finitely many principal congruence manifolds M_z^D are link complements.

The case of discriminant $D = -3$

$$\mathrm{M}_z^{-3} = \frac{\mathbb{H}^3}{\ker\left(\mathrm{PGL}\,(2, \mathbb{Z}[\zeta]) \to \mathrm{PGL}\left(2, \frac{\mathbb{Z}[\zeta]}{\langle z \rangle}\right)\right)}$$

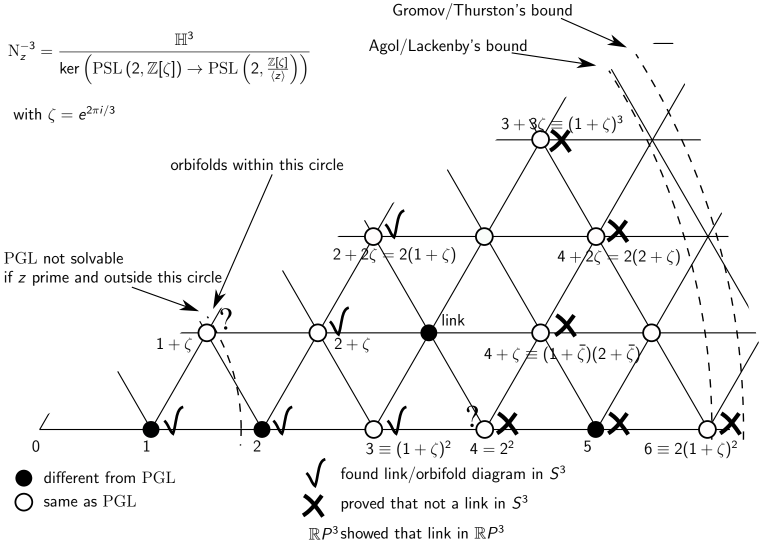
with $\zeta = e^{2\pi i/3}$



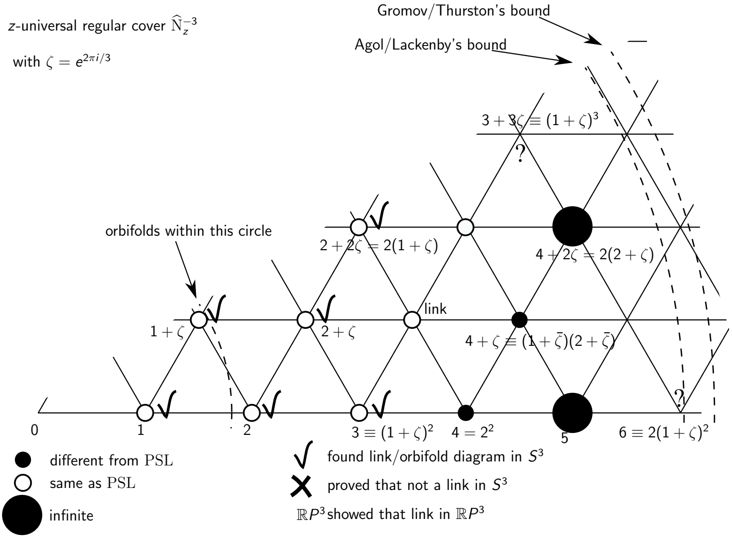
The case of discriminant $D = -3$

$$N_z^{-3} = \frac{\mathbb{H}^3}{\ker\left(\mathrm{PSL}(2,\mathbb{Z}[\zeta]) \to \mathrm{PSL}\left(2, \frac{\mathbb{Z}[\zeta]}{\langle z \rangle}\right)\right)}$$

with $\zeta = e^{2\pi i/3}$



The case of discriminant $D = -3$



Preliminaries: Orbifolds

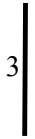
- 3-orbifold M locally modeled on quotient

$$\frac{\mathbb{R}^3}{\Gamma} \to U \subset M$$

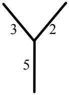
  by a finite subgroup $\Gamma \subset \mathrm{SO}(3, \mathbb{R})$.
- Here, 3-orbifolds M are oriented.
- Underlying topological space $X(M)$ is a 3-manifold.
- Singular locus $\Sigma(M)$ is the set where Γ is non-trivial.
  $\Sigma(M)$ is embedded trivalent graph with labeled edges.
- Near edges of $\Sigma(M)$: modeled on branched cover, Γ cyclic.
- Near vertices of $\Sigma(M)$: Γ is dihedral or orientation-preserving symmetries of a Platonic solid.

Orbifold notation

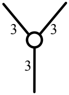
Cusp
(remove knot)



Edge of singular graph
(modeled on branched cover)

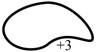


Vertex of singular graph
(modeled on orientation-preserving triangle group)



Cusp of orbifold
(remove a small ball of underlying topological manifold)

$S^1 \subset S^3$ consisting of ∞ and line perpendicular to paper plane



Surgery on knot

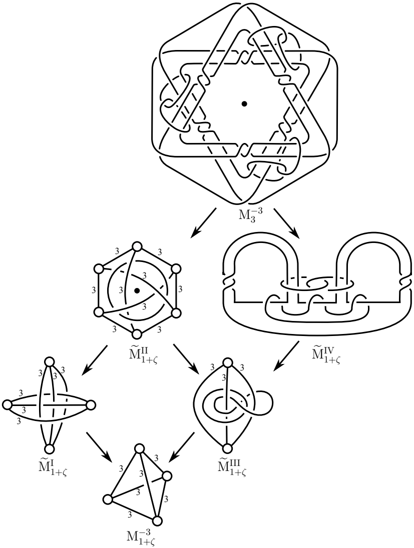
Construction of M_3^{-3}

- M_3^{-3} has 54 regular ideal tetrahedra and 12 cusps.
- The orientation-preserving symmetries are $\mathrm{PGL}\left(2, \frac{\mathbb{Z}[\zeta]}{\langle 3\rangle}\right)$
- **Lemma:** $\mathrm{M}_3^{-3} \to \mathrm{M}_{1+\zeta}^{-3}$ is the universal abelian cover of $\mathrm{M}_{1+\zeta}^{-3}$.
- **Lemma:** The holonomy of this cover is given by

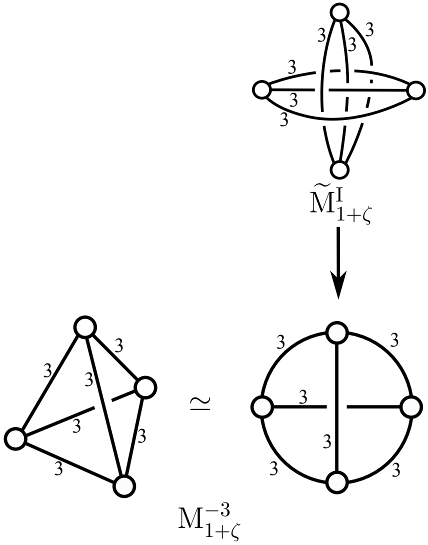
$$\pi_1^{orb}\left(\mathrm{M}_{1+\zeta}^{-3}\right) \twoheadrightarrow \left(\frac{\mathbb{Z}}{3}\right)^3.$$

Reason: $\langle 3\rangle = \langle 1+\zeta\rangle^2$ and $\frac{\mathbb{Z}[\zeta]}{\langle 1+\zeta\rangle} \cong \mathbb{Z}/3$.

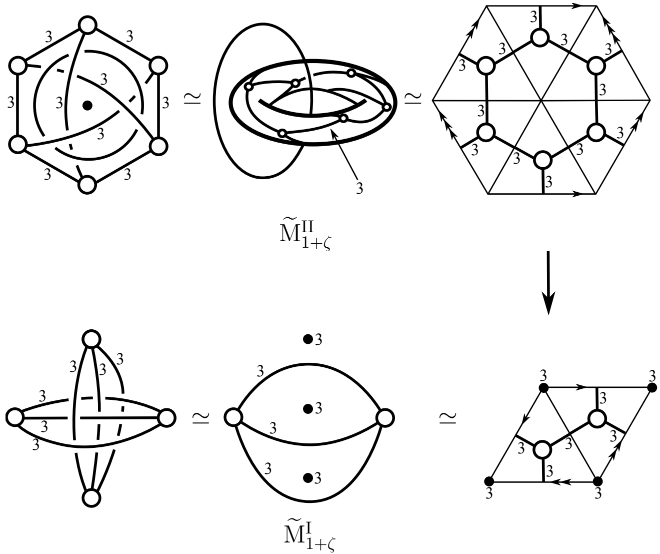
Overview of construction of M_3^{-3}



Step 1 of M_3^{-3}: 3-cyclic cover along unknot



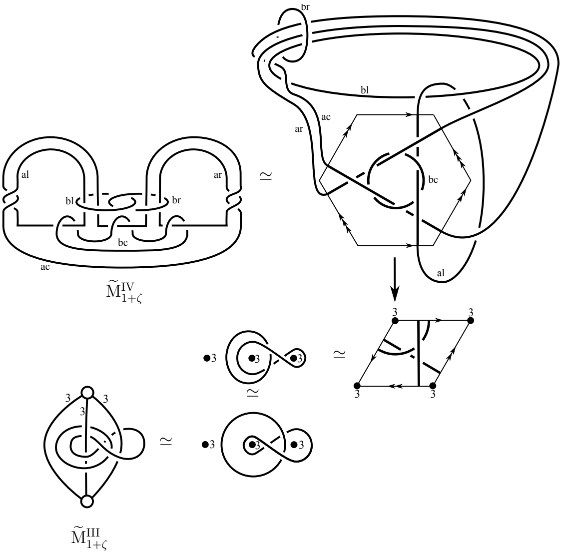
Step 2 of M_3^{-3}: 3-cyclic cover of $(3,3,3)$-triangle orbifold



Step 3 of M_3^{-3}: Divide out 3-cyclic symmetry

- The singular locus is too complicated to construct a 3-cyclic cover.
- Divide out 3-cyclic symmetry.

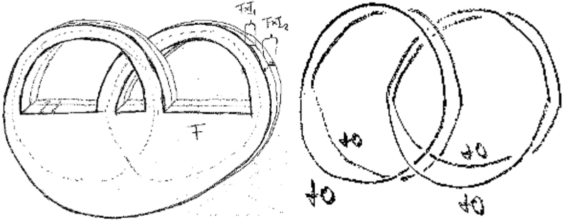
Step 4 of M_3^{-3}: 3-cyclic cover of $(3, 3, 3)$-triangle orbifold



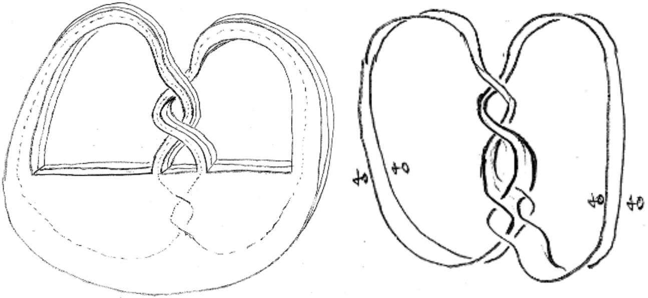
Step 5 of M_3^{-3}: Cover according to Akbulut and Kirby

- Akbulut and Kirby, "Branched Covers of Surfaces in 4-Manifolds": Construction of cyclic cover of B^4 branched over Seifert surface of a link in $S^3 = \partial B^4$ pushed into B^4.
- Here, we are only interested in what happens on the boundary S^3.
- The Seifert surface will determine the holonomy of the cyclic cover branched over a link in S^3.

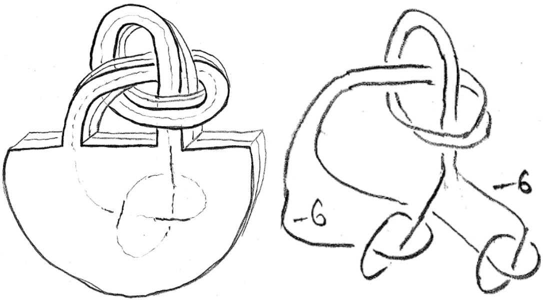
Example of a cyclic cover



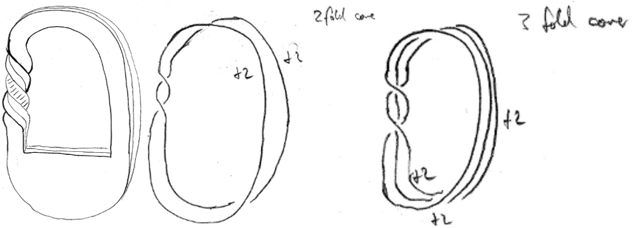
Example of a cyclic cover



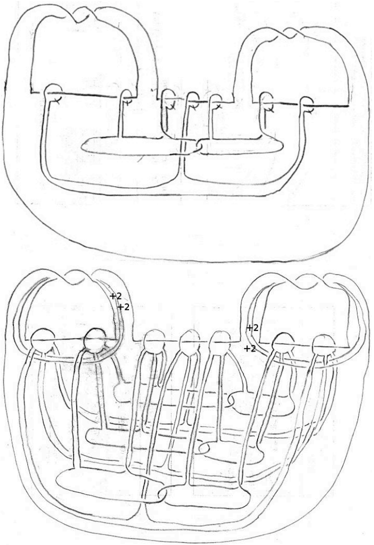
Example of a cyclic cover



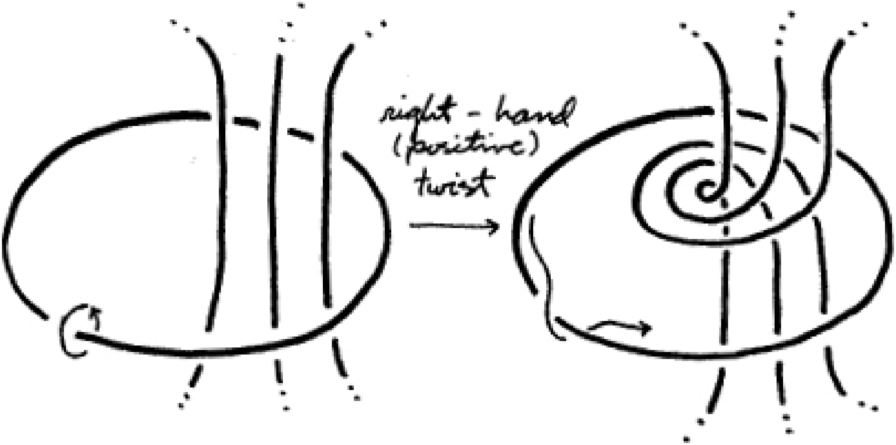
Example of a cyclic cover



Step 5 of M_3^{-3}: Cover according to Akubulut and Kirby

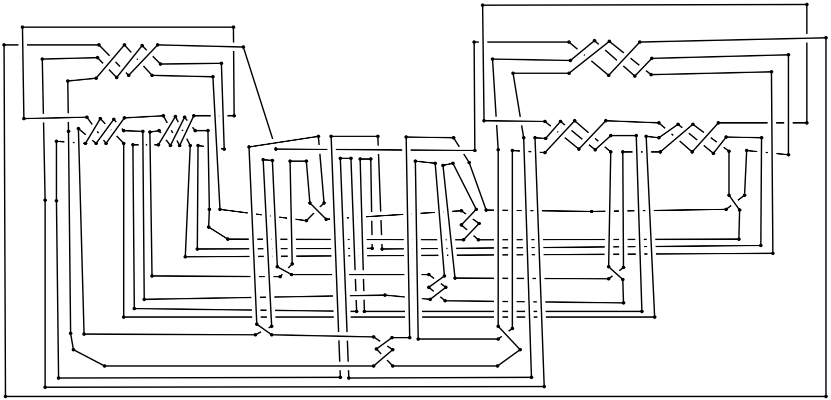


Rolfsen twists

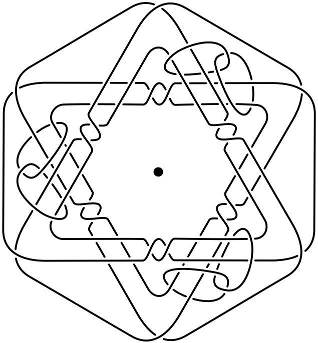


(Source: Rolfsen, Knots and Links)

Step 5 of M_3^{-3}: Rolfsen twists and blow-downs



Dihedral symmetry of link for M_3^{-3}



M_3^{-3}

Construction of $\mathrm{M}^{-3}_{2+2\zeta}$

- $\mathrm{M}^{-3}_{2+2\zeta}$ has 120 regular ideal tetrahedra and 20 cusps.
- Orientation-preserving symmetries are

$$\begin{aligned}\mathrm{PGL}\left(2, \frac{\mathbb{Z}[\zeta]}{\langle 2+2\zeta\rangle})\right) &\cong \mathrm{PGL}\left(2, \frac{\mathbb{Z}[\zeta]}{\langle 1+\zeta\rangle}\right) \oplus \mathrm{PGL}\left(2, \frac{\mathbb{Z}[\zeta]}{\langle 2\rangle}\right) \\ &\cong S_4 \oplus A_5.\end{aligned}$$

- For $G \subset S_4 \oplus A_5$, let

$$|G| = \frac{\mathrm{M}^{-3}_{2+2\zeta}}{G}$$

.

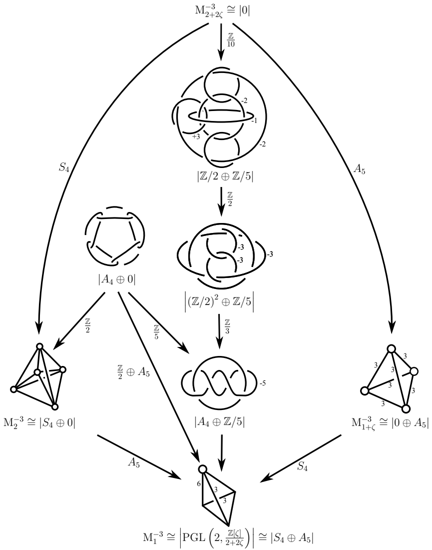
Construction of $\mathrm{M}^{-3}_{2+2\zeta}$

- Orbifold M^{-3}_{2} and manifold double-cover in:
  Dunfield, Thurston, "The virtual Haken conjecture: experiments and examples"
- Decktransformation group of

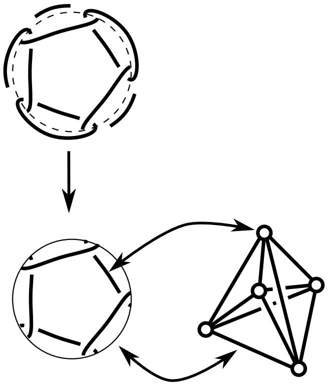
$$\mathrm{M}^{-3}_{2+2\zeta} \cong |0| \to |S_4 \oplus 0| \cong \mathrm{M}^{-3}_{2}$$

  is S_4, a solvable group.
- S_4 and $\mathbb{Z}/5 \subset A_5$ commute in $S_4 \oplus A_5$.
  Can divide 5-cyclic symmetry and postpone 5-cyclic cover until later.

Overview of the construction of $\mathrm{M}^{-3}_{2+2\zeta}$



Pentacle

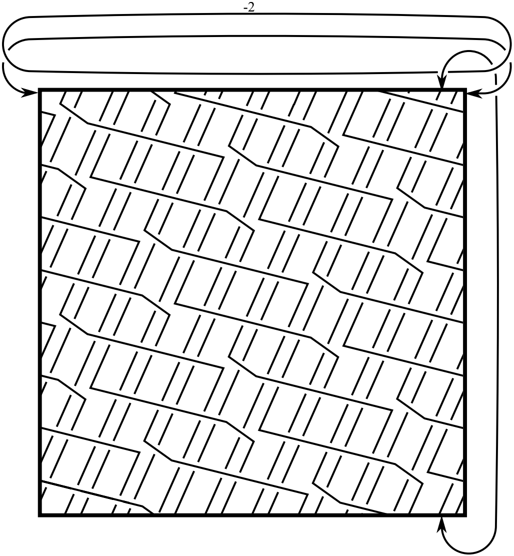


$$\text{Pentacle orbifold} = \frac{\text{Minimally twisted 5-component chain link}}{\text{involution around dotted}}$$

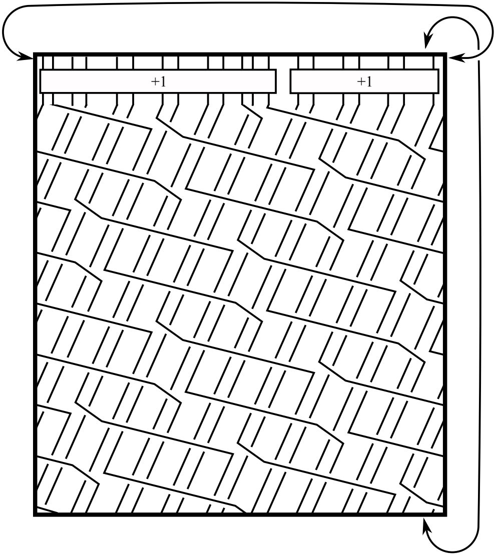
Tricks for $\mathrm{M}_{2+2\zeta}^{-3}$

- Blow-up makes 5-cyclic symmetry of chain link visible.
- Rolfsen twists produce surgery unknots with coefficients $\frac{a}{b}$ with $p|b$. These unknots serve as branching locus for Akbulut and Kirby construction.
- Reduce rational plumbing diagrams to single surgery unknot revealing lens space structure.
- Projection onto torus for visualization.

$\mathrm{M}^{-3}_{2+2\zeta}$ in $\mathbb{R}P^3$



$\mathrm{M}^{-3}_{2+2\zeta}$ in S^3



Progress on the missing links

$z = 3+\zeta, 3+2\zeta, 5+\zeta$ is prime.
For $z = 3+\zeta$:

- Let $G = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$.
- Triangulation of $M = \mathbb{H}^3/p^{-1}(G)$ (Python script).
- $\mathrm{M}^{-3}_{3+\zeta}$ is unique (as manifold) 13-cyclic cover of M with 14 cusps.
- M obtained by $\frac{14}{3}$ Dehn filling of 10^3_{65}.

Open questions

- Find remaining 5 potential principal congruence links, or show manifolds are not link complements.
- Is PGL or PSL more natural?
- Are there infinitely many congruence links?
- Are there infinitely many regular Bianchi orbifold cover links?

Classification of regular Binachi orbifold covers for $D = -3$

Invariant of regular Bianchi orbifold cover: Cusp shape z.
Triangulation by regular tetrahedra induces lattice $\mathbb{Z}[\zeta] \subset \mathbb{C}$ on cusps.
Cusp torus is $\mathbb{C}/\langle z \rangle$ for some $z \in \mathbb{Z}[\zeta]$ determined up to unit.

Fix z. Category of regular Bianchi orbifold covers:

- Finite-volume initial object for

$$z \in \{2, 2+\zeta, 2+2\zeta, 3, 3+\zeta, 3+2\zeta, 4, 4+\zeta\}.$$

- Terminal object is M_z^{-3} for

$$z \in \{2+\zeta, 3+\zeta\}.$$

For the lower z, we have already seen all regular Bianchi orbifold covers.