Principal Congruence Links for Discriminant $D = -3$

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Overview

- Thurston congruence link, geometric description
- Bianchi orbifolds, congruence and principal congruence manifolds
- Results implying there are finitely many principal congruence links
- Overview for the case of discriminant $D = -3$
- Preliminaries for the construction
- Construction of two more examples
- Open questions
Complement is non-compact finite-volume hyperbolic 3-manifold.

Tesselated by 28 regular ideal hyperbolic tetrahedra.

Tesselation is “regular”, i.e., symmetry group takes every tetrahedron to every other tetrahedron in all possible 12 orientations.
Cusped hyperbolic 3-manifolds

- Ideal hyperbolic tetrahedron does not include the vertices.
- Remove a small horoball. Ideal tetrahedron is topologically a truncated tetrahedron.
- Cut is a triangle with a Euclidean structure from hororsphere.
Cusped hyperbolic 3-manifolds have toroidal ends

- Truncated tetrahedra form interior of a 3-manifold $\tilde{M}$ with boundary.
- $\partial \tilde{M}$ triangulated by the Euclidean triangles.
- $\partial \tilde{M}$ is a torus.
- Ends (cusps) of hyperbolic manifold modeled on torus $\times$ interval.
Knot complements can be cusped hyperbolic 3-manifolds

- Cusp homeomorphic to a tubular neighborhood of a knot/link component.
- Figure-8 knot complement tesselated by two regular ideal tetrahedra.
- Hyperbolic metric near knot so dense that light never reaches knot.
- Complement still has finite volume.
“Regular tesselations”

- Spherical 2-dimensional version of “regular tesselations”: Platonic solids.
- Person in a tile cannot tell through intrinsic measurements in what tile he or she is or at what edge he or she is looking at.
Two more examples

$M_{3}^{-3}$

54 regular ideal tetrahedra

$M_{2+2\zeta}^{-3}$

120 regular ideal tetrahedra
Thurston congruence link and the Klein quartic

\[ xy^3 + yz^3 + zx^3 = 0 \]

- Faces of ideal tetrahedra form immersed hyperbolic surface.
- Filling the punctures yields an algebraic curve in \( \mathbb{C}P^2 \): Klein quartic.
- Orientation-preserving symmetry group of the hyperbolic surface: \( \text{PSL}(2,7) \), the unique finite simple group of order 168.
- Thurston/Agol, “Thurston congruence link”
Bianchi orbifolds

- $\mathcal{O}_D$: ring of integers in $\mathbb{Q}(\sqrt{D})$. $D < 0$, $D \equiv 0, 1(4)$ discriminant.
- **Bianchi group**:
  
  $\text{PGL}(2, \mathcal{O}_D)$ respectively $\text{PSL}(2, \mathcal{O}_D)$

  is a discrete subgroup of $\text{PGL}(2, \mathbb{C}) \cong \text{PSL}(2, \mathbb{C}) \cong \text{Isom}^+(\mathbb{H}^3)$.

- **Bianchi orbifold**:
  
  $M_1^D = \frac{\mathbb{H}^3}{\text{PGL}(2, \mathcal{O}_D)}$ respectively $\frac{\mathbb{H}^3}{\text{PSL}(2, \mathcal{O}_D)}$.

- Every cusped arithmetic hyperbolic manifold is commensurable with a Bianchi orbifold.
Bianchi orbifolds

$M_{1}^{-3}$

regular ideal tetrahedron divided by orientation-preserving symmetries

$M_{1}^{-4}$

regular ideal octahedron divided by orientation-preserving symmetries
Congruence subgroups

- Fix ideal $I$ in $\mathcal{O}_D$.
- $\mathcal{O}_D \rightarrow \mathcal{O}_D/I$ induces map
  
  $$p : \text{PGL}(2, \mathcal{O}_D) \rightarrow \text{PGL}(2, \mathcal{O}_D/I)$$

- **Congruence subgroup:**
  
  $p^{-1}(G)$ for some subgroup $G \subset \text{PGL}(2, \mathcal{O}_D/I)$.

- **Principal congruence subgroup:**
  
  $$\ker(p) = p^{-1}(0).$$

- **(Principal) congruence manifold/orbifold:** quotient of $\mathbb{H}^3$
  
  $$\mathbb{M}_z^D = \frac{\mathbb{H}^3}{\ker \left( \text{PGL} \left( 2, \mathcal{O}_D \right) \rightarrow \text{PGL} \left( 2, \frac{\mathcal{O}_D}{\langle z \rangle} \right) \right)}$$

- Thurston congruence link complement is $\mathbb{M}_{2+\zeta}^{-3}$. 
Cuspidal Cohomology, Baker’s links

- Cuspidal cohomology yields an obstruction:
  If $M^D_1$ can be covered by a link complement, then $D \in \mathcal{L}$ where
  $$\mathcal{L} = \{-3, -4, -7, -8, -11, -15, -19, -20, -23, -24, -31, -39, -47, -71\}.$$  

- Mark Baker constructed “some” cover for each $D \in \mathcal{L}$, making it ‘iff’.
  - His covers are neither canonical nor regular.
Finitely many principal congruence links

- **Gromov and Thurston $2\pi$-Theorem:** Dehn filling cusps of a hyperbolic manifold along peripheral curves with length $> 2\pi$ yields hyperbolic manifold again.
  (Length measured on embedded horoballs)

- **Agol and Lackenby:** improved bound to $> 6$.

- **Corollary:** If the shortest curve on every cusp has length $> 6$, the manifold is not a link complement.

- Hence, only finitely many principal congruence manifolds $M^D_z$ are link complements.
The case of discriminant $D = -3$

\[ M_z^{-3} = \frac{\mathbb{H}^3}{\ker(PGL(2, \mathbb{Z}[\zeta]) \to PGL(2, \mathbb{Z}[\zeta]))} \]

with $\zeta = e^{\frac{2\pi i}{3}}$

Gromov/Thurston’s bound
Agol/Lackenby’s bound

orbifolds within this circle

PGL not solvable if $z$ prime and outside this circle

found link/orbifold diagram in $S^3$
proved that not a link in $S^3$
$\mathbb{R}P^3$ showed that link in $\mathbb{R}P^3$
The case of discriminant $D = -3$

$$N_z^{-3} = \frac{\mathbb{H}^3}{\ker(\text{PSL}(2, \mathbb{Z}[\zeta]) \to \text{PSL}(2, \mathbb{Z}[\frac{1}{z}]))}$$

with $\zeta = e^{2\pi i/3}$

Gromov/Thurston’s bound

Agol/Lackenby’s bound

PGL not solvable if $z$ prime and outside this circle

orbifolds within this circle

$3 + 8\zeta \equiv (1 + \zeta)^3$

$2 + 2\zeta = 2(1 + \zeta)$

$4 + 2\zeta = 2(2 + \zeta)$

$4 + \zeta \equiv (1 + \bar{\zeta})(2 + \bar{\zeta})$

$2 + \zeta$

$3 \equiv (1 + \zeta)^2$

$4 = 2^2$

$5$

$6 \equiv 2(1 + \zeta)^2$

different from PGL

same as PGL

√ found link/orbifold diagram in $S^3$

× proved that not a link in $S^3$

$\mathbb{R}P^3$ showed that link in $\mathbb{R}P^3$
The case of discriminant $D = -3$

$z$-universal regular cover $\hat{N}_z^{-3}$

with $\zeta = e^{2\pi i / 3}$

orbifolds within this circle

Gromov/Thurston's bound

Agol/Lackenby's bound

$3 + 3\zeta \equiv (1 + \zeta)^3$

$2 + 2\zeta = 2(1 + \zeta)$

$2 + \zeta$

$4 + 2\zeta = 2(2 + \zeta)$

$4 + \zeta \equiv (1 + \zeta)(2 + \zeta)$

link

$0$

$1$

$2$

$3 \equiv (1 + \zeta)^2$

$4 = 2^2$

$5$

$6 \equiv 2(1 + \zeta)^2$

- different from PSL
- same as PSL
- infinite

$\sqrt{\text{found link/orbifold diagram in } S^3}$

$\times \text{proved that not a link in } S^3$

$\mathbb{R}P^3$ showed that link in $\mathbb{R}P^3$
Preliminaries: Orbifolds

- 3-orbifold $M$ locally modeled on quotient
  $\frac{\mathbb{R}^3}{\Gamma} \rightarrow U \subset M$

  by a finite subgroup $\Gamma \subset \text{SO}(3, \mathbb{R})$.

- Here, 3-orbifolds $M$ are oriented.

- Underlying topological space $X(M)$ is a 3-manifold.

- Singular locus $\Sigma(M)$ is the set where $\Gamma$ is non-trivial.
  $\Sigma(M)$ is embedded trivalent graph with labeled edges.

- Near edges of $\Sigma(M)$: modeled on branched cover, $\Gamma$ cyclic.

- Near vertices of $\Sigma(M)$: $\Gamma$ is dihedral or orientation-preserving
  symmetries of a Platonic solid.
Orbifold notation

- **Cusp** (remove knot)
- Edge of singular graph (modeled on branched cover)
- Vertex of singular graph (modeled on orientation-preserving triangle group)
- Cusp of orbifold (remove a small ball of underlying topological manifold)

$S^1 \subset S^3$ consisting of $\infty$ and line perpendicular to paper plane

- Surgery on knot
Construction of $M_3^{-3}$

- $M_3^{-3}$ has 54 regular ideal tetrahedra and 12 cusps.
- The orientation-preserving symmetries are $\text{PGL} \left( 2, \frac{\mathbb{Z}[[\zeta]]}{\langle 3 \rangle} \right)$
- **Lemma:** $M_3^{-3} \to M_1^{-3} \zeta$ is the universal abelian cover of $M_1^{-3} \zeta$.
- **Lemma:** The holonomy of this cover is given by

$$\pi_1^{\text{orb}} \left( M_1^{-3} \zeta \right) \to \left( \frac{\mathbb{Z}}{3} \right)^3.$$

**Reason:** $\langle 3 \rangle = \langle 1 + \zeta \rangle^2$ and $\frac{\mathbb{Z}[[\zeta]]}{\langle 1 + \zeta \rangle} \cong \mathbb{Z}/3$. 
Overview of construction of $M_3^{−3}$

Abelian covers of the Bianchi orbifold for $O^{−3}$ involved in the construction of $M_3^{−3}$. Each arrow is a 3-cyclic cover.
Step 1 of $M_3^{-3}$: 3-cyclic cover along unknot

\[ M_3^{-3} \approx \tilde{M}_{1+\zeta} \]

\[ \tilde{M}_{1+\zeta} \approx M_1^{-3} \]
Step 2 of $\mathbb{M}_3^{-3}$: 3-cyclic cover of $(3, 3, 3)$-triangle orbifold

$\mathbb{M}_1^{\Pi+\zeta}$

$\mathbb{M}_1^{I+\zeta}$

$\mathbb{M}_1^{I}$

$\mathbb{M}_1^{I+\zeta}$

Figure 1.7: Construction of $\tilde{\mathbb{M}}_1^{I+\zeta}$.
Step 3 of $M_3^{-3}$: Divide out 3-cyclic symmetry

- The singular locus is too complicated to construct a 3-cyclic cover.
- Divide out 3-cyclic symmetry.
Step 4 of $\mathbb{M}_3^{-3}$: 3-cyclic cover of $(3, 3, 3)$-triangle orbifold
Step 5 of $M_3^{-3}$: Cover according to Akbulut and Kirby

- Akbulut and Kirby, “Branched Covers of Surfaces in 4-Manifolds”: Construction of cyclic cover of $B^4$ branched over Seifert surface of a link in $S^3 = \partial B^4$ pushed into $B^4$.
- Here, we are only interested in what happens on the boundary $S^3$.
- The Seifert surface will determine the holonomy of the cyclic cover branched over a link in $S^3$. 
Example of a cyclic cover
Example of a cyclic cover
Example of a cyclic cover
Example of a cyclic cover
Step 5 of $M_3^{-3}$: Cover according to Akubulut and Kirby

Figure 1.10: Construction of 3-cyclic branched cover $M_3^{-3} \to \tilde{M}_{IV}^{1+\zeta}$. The resulting link representation of $M_3^{-3}$ has four $+2$ surgeries.
Rolfsen twists

(Source: Rolfsen, Knots and Links)
Step 5 of $M_3^{-3}$: Rolfsen twists and blow-downs
Dihedral symmetry of link for $M_3^{-3}$
Construction of $M_{2+2\zeta}^{-3}$

- $M_{2+2\zeta}^{-3}$ has 120 regular ideal tetrahedra and 20 cusps.
- Orientation-preserving symmetries are
  \[
  PGL \left( 2, \frac{\mathbb{Z}[\zeta]}{\langle 2 + 2\zeta \rangle} \right) \cong PGL \left( 2, \frac{\mathbb{Z}[\zeta]}{\langle 1 + \zeta \rangle} \right) \oplus PGL \left( 2, \frac{\mathbb{Z}[\zeta]}{\langle 2 \rangle} \right) \cong S_4 \oplus A_5.
  \]
- For $G \subset S_4 \oplus A_5$, let
  \[
  |G| = \frac{M_{2+2\zeta}^{-3}}{G}
  \]
Construction of $M_{2+2\zeta}^{-3}$

- Orbifold $M_{2}^{-3}$ and manifold double-cover in:
  Dunfield, Thurston, “The virtual Haken conjecture: experiments and examples”

- Decktransformation group of

  $$M_{2+2\zeta}^{-3} \cong |0| \rightarrow |S_4 \oplus 0| \cong M_{2}^{-3}$$

  is $S_4$, a solvable group.

- $S_4$ and $\mathbb{Z}/5 \subset A_5$ commute in $S_4 \oplus A_5$.
  Can divide 5-cyclic symmetry and postpone 5-cyclic cover until later.
Overview of the construction of $M_{2+2\zeta}^{-3}$
Pentacle orbifold = \[ \text{Minimally twisted 5-component chain link} \]
\[ \text{involution around dotted} \]
Blow-up makes 5-cyclic symmetry of chain link visible.

Rolfsen twists produce surgery unknots with coefficients $\frac{a}{b}$ with $p \mid b$. These unknots serve as branching locus for Akbulut and Kirby construction.

Reduce rational plumbing diagrams to single surgery unknot revealing lens space structure.

Projection onto torus for visualization.
$M_{2+2\zeta}^{-3}$ in $\mathbb{RP}^3$
$M_{2+2\zeta}^{-3}$ in $S^3$
Progress on the missing links

\[z = 3 + \zeta, 3 + 2\zeta, 5 + \zeta \text{ is prime.}\]

For \(z = 3 + \zeta\):

- Let \(G = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\} \).
- Triangulation of \(M = \mathbb{H}^3/p^{-1}(G)\) (Python script).
- \(M_{3+\zeta}^{-3}\) is unique (as manifold) 13-cyclic cover of \(M\) with 14 cusps.
- \(M\) obtained by \(\frac{14}{3}\) Dehn filling of \(10_65^3\).
Open questions

- Find remaining 5 potential principal congruence links, or show manifolds are not link complements.
- Is PGL or PSL more natural?
- Are there infinitely many congruence links?
- Are there infinitely many regular Bianchi orbifold cover links?
Invariant of regular Bianchi orbifold cover: Cusp shape $z$.
Triangulation by regular tetrahedra induces lattice $\mathbb{Z}[\zeta] \subset \mathbb{C}$ on cusps.
Cusp torus is $\mathbb{C}/\langle z \rangle$ for some $z \in \mathbb{Z}[\zeta]$ determined up to unit.

Fix $z$. Category of regular Bianchi orbifold covers:

- Finite-volume initial object for

$$z \in \{2, 2 + \zeta, 2 + 2\zeta, 3, 3 + \zeta, 3 + 2\zeta, 4, 4 + \zeta\}.$$

- Terminal object is $M_{z^{-3}}$ for

$$z \in \{2 + \zeta, 3 + \zeta\}.$$

For the lower $z$, we have already seen all regular Bianchi orbifold covers.