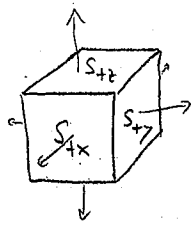
Solutions to Worksheet on Surface Integrals 04/30/08

1.



box has 6 surfaces

S_{+x} $x=+1$
S_{-x} $x=-1$
S_{+y}
S_{-y}
S_{+z}
S_{-z}

By symmetry $\iint_{S_{+x}} F\cdot dS = \iint_{S_{-x}} F\cdot dS = \iint_{S_{+y}} --$

hence $\iint_S F\cdot dS = 6 \iint_{S_{+x}} F\cdot dS$

S_{+x} parametrized by $r(u,v) = (u,v,1)$ where $-1\le u,v\le 1$

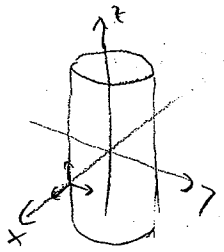
$dS = r_u \times r_v \, du\, dv$
$= (0,0,1)\, du\, dv$
$F(r(u,v)) = (u,v,1)$

$$\iint_{S_{+x}} F\cdot dS = \int_{-1}^{+1}\int_{-1}^{+1} (u,v,1)\cdot(0,0,1)\, du\, dv = \iint_{-1\le u,v\le 1} du\, dv = 4$$

$$\iint_S F\cdot dS = 6\cdot 4 = \boxed{24}$$

Divergence Thm: $\iiint_{-1\le x,y,z\le 1} \operatorname{div} F\, dx\, dy\, dz = \iiint_{-1\le x,y,z\le 1} 3\, dx\, dy\, dz = \boxed{2^3\, 3}$

2.



$r(u,v) = (\cos u, \sin u, v)$

$0 \le u \le 2\pi$

$dS = r_u \times r_v \, du\, dv = (\cos u, \sin u, 0)$

$$F(r(u,v)) = \frac{(\cos u, \sin u, v)}{\sqrt{\cos^2 u + \sin^2 u + v^2}^4} = \frac{(\cos u, \sin u, v)}{\sqrt{1+v^2}^4}$$

$$F(r(u,v))\cdot dS = \frac{1}{\sqrt{1+v^2}^4} = \frac{1}{(1+v^2)^2}$$

$$\iint_S F\cdot dS = \int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{1}{(1+v^2)^2}\, du\, dv = 2\pi \int_{-\infty}^{+\infty} \frac{1}{(1+v^2)^2}\, dv$$

$v = \tan\theta$, $dv = \sec^2\theta\, d\theta$

$$2\pi \int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \boxed{2\pi^2}$$

3. Surface parametrized by $r(u,v) = (u,v,1-u-v)$ $-1\le u,v\le +1$

$dS = |dS| = |r_u \times r_v|\, du\, dv = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = |i+j+k|\, du\, dv = \sqrt{3}\, du\, dv$

$$\iint_S dS = \int_{-1}^{+1}\int_{-1}^{+1} \sqrt{3}\, du\, dv = \boxed{4\sqrt{3}}$$

4. S parametrized by $r(u,v) = (u, v, 0)$ $\quad u^2+v^2 \le 1$

$$dS = r_u \times r_v \, du\,dv = k\, du\, dv$$

$$F \cdot dS = \frac{du\,dv}{u^2+v^2+1}$$

$$\iint_S F\,dS = \int_{-1}^{+1}\int_{-\sqrt{1-v^2}}^{+\sqrt{1-v^2}} \frac{du\,dv}{u^2+v^2+1} = \int_0^{2\pi}\int_0^1 \frac{r\,d\theta\,dr}{r^2+1} = 2\pi \int_0^1 \frac{2r\,dr}{r^2+1} = \pi \int_0^1 \frac{du}{u+1}$$

polar coordinates

$u = r^2$

$du = 2r\,dr$

$= \pi(\ln 2)$