Characterization of Riemann integrable functions

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1 Statement

Definition 1. *A set $U \subset \mathbb{R}^n$ has Lebesgue measure 0 if for every $\epsilon > 0$ there is a countable cover of U by open/closed rectangles such that the total volume is less than ϵ.*

Theorem 1. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is Riemann-integrable if and only if it is*

- *bounded*
- *has compact support and*
- *the set $K = \{x | f \text{ not continuous at } x\}$ has Lebesgue-measure 0.*

Before we prove this theorem, recall the following definition:

Definition 2. *A set $U \subset \mathbb{R}^n$ is compact if every open cover of U (i.e., a collection of open sets U_i such that $U \subset \bigcup_{i \in I} U_i$) has a finite subcover.*

Theorem 2. *A set $U \subset \mathbb{R}^n$ is compact if and only if it is sequentially compact.*

2 Proof for the "if" direction

Let $S = [-2^m, 2^m]^n$ be large enough to contain the support of f, in particular, $f(x) = 0$ for x not in S.
Let M be a bound for f such that $|f(x)| < M$ for all x in $\mathbb{R}^n$.
Let K be as in the theorem.

Given $\epsilon > 0$, we want to show that there is a k such that

$$G = U(f, P_k) - L(f, P_k) < \epsilon. \tag{1}$$

Since K has Lebesgues-measure 0, we can cover K by open rectangles $I_1, I_2, \dots$ such that

$$\sum \text{vol}(I_i) < \frac{\epsilon}{8M}. \tag{2}$$

Since f is continuous, we can pick for each point x in $S \setminus K$ an open rectangle I'_x such that

$$|f(q) - f(x)| < \frac{\epsilon}{8n\text{vol}(S)}. \tag{3}$$

All I'_x will cover the complement of $S \setminus K$ and all I_i cover K, so together they form a cover $\{I_i, I'_x\}$ of S. But S is compact, so we can pick a finite subcover consisting of some rectangles of both of the above types, say, $I_1, \ldots, I_s, I'_{x_1}, \ldots, I'_{x_t}$.

Let k_1 be large enough such that

$$U(\chi_{I_1} + \cdots + \chi_{I_p}, P_{k_1}) < \frac{\epsilon}{4M}. \tag{4}$$

This is possible because $\chi_{I_1} + \cdots + \chi_{I_p}$ is Riemann-integrable (being a finite sum of characteristic functions of rectangles, we proved this earlier) and, by (2), the integral is less than $\epsilon/(8M)$. So, we can pick k_1 such that $U(\chi_{I_1} + \cdots + \chi_{I_p}, P_{k_1}) - \int \chi_{I_1} + \cdots + \chi_{I_p} < \epsilon/(8M)$.

Write a rectangle I'_{x_j} as $(a_{1,j}, b_{1,j}) \times \cdots \times (a_{n,j}, b_{n,j})$. Let X be the set of all $a_{i,j}$ and $b_{i,j}$. Notice that X is finite. We can pick k_2 large enough so that no two numbers in X fall into the same interval of the standard partition P_{k_2} of $\mathbb{R}$.

Now, let k be larger than k_1 and k_2. We claim that $G < \epsilon$. To see this, divide the rectangles I in P_k into "bad" rectangles that intersect some I_i and "good" rectangles that intersect no I_i.

On a bad rectangle, the difference of the supremum and infimum of f is at most $2M$, so by (4), the total contribution of the bad rectangles to G is bounded by $2M \frac{\epsilon}{4M} = \frac{\epsilon}{2}$.

Let q and q' be two points in I'_{x_j}. Then, both $|f(x) - f(q)|$ and $|f(x) - f(q')|$ are less than $\frac{\epsilon}{8n\text{vol}(S)}$, so $|f(q) - f(q')| < \frac{\epsilon}{4n\text{vol}(S)}$.

Pick q and q' in a good rectangle in P_k such that the line segment is parallel to one of the coordinate axis. Then, by the condition on k_2, the line segment can be covered by at most two rectangles I'_{x_j}, and thus $|f(q) - f(q')| < \frac{\epsilon}{2n\text{vol}(S)}$.

Pick arbitrary q and q' in a good rectangle in P_k. Then, there is a path consisting of n line segments connecting the two points such that each line segment is parallel to a coordinate axis. That means that $|f(q) - f(q')| < \frac{\epsilon}{2\text{vol}(S)}$. Hence, the difference of the supremum and infimum of f on a good rectangle is at most $\frac{\epsilon}{2\text{vol}(S)}$, and the total contribution to G is at most $\frac{\epsilon}{2}$.

On rectangles outside of S, f is zero and there is no contribution to $U(f, P_k) - L(f, P_k)$. Adding up the contributions to G by the good and bad rectangles we get at most $\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Thus, we have shown that f is Riemann-integrable.