**There are more problems on this exam than you need to solve. From the problems below pick a subset worth exactly 180 points and solve them. Clearly indicate which problems you picked. You will be graded on these problems. Do not pick problems totaling more than 180 points.**

*To receive full credit, you must show all of your work. No books, notes or electronic equipment are allowed. You have* 50 *minutes to complete the exam.*

*Please write your name (last, first) and UID at the top right of each page.*

Good luck!

15 P. 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Let $p = (1, 1)$. Suppose that $\frac{\partial f}{\partial x}(0, 0) = 2$, $\frac{\partial f}{\partial p}(1, 0) = 1$, $\frac{\partial f}{\partial p}(0, 0) = 3$. Compute $\frac{\partial f}{\partial y}(0, 0)$.

15 P. 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x\cos(y)$. Give a formula for the tangent plane at the point $(1, \pi)$.

30 P. 3. Consider $\mathbb{R}^n$. Is there a collection of closed sets $\{U_i\}$ in $\mathbb{R}^n$ such that their union, respectively, their intersection is the open unit ball $\mathcal{B}_1(0)$? For each case (union, intersection), either give an example or proof that it is impossible.

30 P. 4. Let $X$ be the set of all continuous functions $f : [0, 1] \to \mathbb{R}$. For which of the following functions $d$ does the set $X$ together with $d$ form a metric space? If $d$ is not a valid metric, give a brief explanation what necessary property of $d$ is violated. Otherwise, no further explanation is necessary.

(a) $d(f, g) = \left|f\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right|$

(b) $d(f, g) = \left|\max_{x\in[0,1]} f(x) - \max_{x\in[0,1]} g(x)\right|$

(c) $d(f, g) = \max_{x\in[0,1]} |f(x) - g(x)|$

(d) $d(f, g) = \max_{x\in[0,1]} (f(x) - g(x))^2$

(e) $d(f, g) = \int_0^1 |f(x) - g(x)|dx$

(f) $d(f, g) = \int_0^1 |f(x)|dx$

60 P. 5. Answer each of the following questions with yes/no/can't tell. No justification necessary.

(a) The set $U$ is not connected. Is $U$ pathwise-connected?

(b) Let $U \subseteq \mathbb{R}^n$ be a convex subset and $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. Is $f(U)$ convex?

(c) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous and $U \subseteq \mathbb{R}^n$ be open. Is $f(U)$ an open set in $\mathbb{R}^m$?

(d) Let $\mathcal{B}_1(0)$ be the open unit ball in $\mathbb{R}^n$ and $f : \mathcal{B}_1(0) \to \mathbb{R}$ be a continuous function. Let $K$ be a subset of $\mathcal{B}_1(0)$ that is closed in $\mathbb{R}^n$. Is $f(K)$ sequentially compact?

(e) Let $U \subseteq \mathbb{R}^n$ be a convex subset and $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. Is $f(U)$ connected?

(f) Let $K \subseteq \mathbb{R}^m$ be a sequentially compact set and $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. Is $f^{-1}(K)$ closed in $\mathbb{R}^n$?

(g) Is every Cauchy sequence in $\mathcal{B}_1(0) \subseteq \mathbb{R}^n$ converging to a point in $\mathcal{B}_1(0)$?

(h) Let $f(x) = \sin(x)$ where $x \in \mathbb{R}$. Is $f$ a contraction?

(i) Let $f(x) = \frac{1}{2}\cos(x)$. Does $f(x)$ have a unique fixed point?

(j) Let $U$ be a closed set in the complete metric space $X$ with metric $d$. Let $\{u_k\}$ be a sequence in $U$ such that $d(u_k, u_{k+1}) = 2^{-k}$. Does $\{u_k\}$ converge to a point in $U$?

(k) Let $\overline{\mathcal{B}_1(0)} = \{x \text{ such that } ||x|| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$. Is there a sequence $\{u_k\}$ in $U$ such that $d(u_k, u_{k+1}) = 1/k$ and $u_k$ converges to a point in $\overline{\mathcal{B}_1(0)}$?

(l) Is there a subset $U \subseteq \mathbb{R}^n$ such that $U$ is not empty and $U$ is not $\mathbb{R}^n$ but the boundary $\text{bd}U$ is open?

15 P. 6. Let $K$ be a sequentially compact subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a continuous function with $f(x) > 0$ for all $x$ in $K$. Prove that there is an $r > 0$ such that $f(x) > r$ for all $x$ in $K$.

15 P. 7. Without explanation, which of the following differential equations has a unique solution for at least some open interval $(-\epsilon, \epsilon)$ with $\epsilon > 0$?

(a) $y'(t) = \cos(y(t))$ with $y(0) = 1$

(b) $y'(t) = y(t)^2$ with $y(0) = 1$

(c) $y'(t) = \sqrt{y(t)}$ with $y(0) = 0$

30 P. 8. Let $X$ and $Y$ be metric spaces. Consider the following conditions on a function $f : X \to Y$:

1. If $V$ is an open set in $Y$, then $f^{-1}(V)$ is open in $X$.
2. For every point $x$ in $X$ and every $\epsilon > 0$, there is a $\delta > 0$ such that for all $w$ in $X$

$$\text{dist}(x, w) < \delta \quad \Rightarrow \quad \text{dist}(f(x), f(w)) < \epsilon.$$

Show that condition 2 implies condition 1.

45 P. 9. Let $U \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$ such that the boundary $\text{bd}U$ is empty. Prove that $U$ is either empty or $\mathbb{R}^n$.