1.)

$$\frac{\partial f}{\partial p}(0,0) = \frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial y}(0,0)$$

$$3 = 2 + \frac{\partial f}{\partial y}(0,0) \qquad \boxed{\frac{\partial f}{\partial y}(0,0) = 1}$$

2.)

$$f(1,\pi) = 1 \cdot \cos\pi = -1$$

$$\frac{\partial f}{\partial x}(x,y) = \cos y \qquad \frac{\partial f}{\partial y}(x,y) = -x\sin y$$

$$\frac{\partial f}{\partial x}(1,\pi) = -1 \qquad \frac{\partial f}{\partial y}(1,\pi) = 0$$

Tangent plane: $f(1,\pi) + (x-1)\frac{\partial f}{\partial x}(1,\pi) + (y-\pi)\frac{\partial f}{\partial y}(1,\pi)$

$$= -1 + (x-1)\cdot(-1) \boxed{= -x}$$

3.)

Let $U_i = \{ x \in \mathbb{R}^n \text{ such that } \|x\| \leq 1 - 1/i \}$. Collection of closed sets

and $\bigcup U_i = B_1^\circ$. (Example for union)

However, the intersection of closed sets is always closed (Basic property of closed sets), and B_1° is not closed. (Proof that impossible for intersection)

4a) $\boxed{\text{No.}}$ $f(x)=x$ $g(x)=\frac{1}{2}$ are two distinct functions with $d(f,g)=0$, so non-negativity violated.

4b) $\boxed{\text{No.}}$ Similary to 4a: $f(\cancel{x})=1-x$ $g(x)=x$ with $d(f,g)=0$ but $f\not\equiv g$.

4c) $\boxed{\text{Yes.}}$ (see book)

4d) $\boxed{\text{No}}$ let $f(x)=0$, $g(x)=1$, $h(x)=2$. Then $d(f,g)=1$, $d(g,h)=1$

$d(f,h)=4$, so

triangle inequality violated.

4e) $\boxed{\text{Yes}}$ (see book)

4f) $\boxed{\text{No.}}$ Let $f(x)=0$, $g(x)=1$. Then $d(f,g)=0$, $d(g,f)=1$,

so symmetry violated.

5a) $\boxed{\text{No.}}$ (Pathwise-connected $\Rightarrow$ connected, so (not connected) $\Rightarrow$ (not pathwise-connected))

5b) $\boxed{\text{Can't tell.}}$

5c) $\boxed{\text{Can't tell.}}$

5d) $K \subset B_1(0)$ is bounded and closed, hence seq. compact, and so is the image $f(K)$. $\boxed{\text{Yes.}}$

5e) U is convex, hence connected, and so is its image $f(U)$. $\boxed{\text{Yes.}}$

5f) K is seq. compact, hence closed, and so is its preimage $f^{-1}(K)$. $\boxed{\text{Yes}}$

5g) Pick $\{(1-1/n, 0, \dots, 0)\} \subseteq B_1(0)$. Sequence is Cauchy and converging to $(1, 0, \dots, 0)$ not in $B_1(0)$. $\boxed{\text{No.}}$

5h) Let $C<1$ be a Lipschitz constant. Then $|\sin(x)-\sin(y)| \leq C|x-y|$, in particular, $|\sin(x)-0| \leq C|x|$, but $\lim_{x\to 0}\left|\frac{\sin(x)}{x}\right| = 1$, so $C<1$ is a contradiction. Hence, f not contraction. $\boxed{\text{No}}$

5i) $|f'(x)| = |-\frac{1}{2}\sin(x)| \leq 1/2$, so f Lipschitz with constant $1/2$, and $\mathbb{R}$ complete, so f is a contraction. $\boxed{\text{Yes.}}$

5j) U closed subset of complete space, so U complete.

$s > t$

$$d(u_s, u_t) \leq \sum_{k=s}^{t-1} d(u_k, u_{k+1}) = \sum_{k=s}^{t-1} \left(\frac{1}{2}\right)^k \leq \left(\frac{1}{2}\right)^{s-1}$$

so $\{u_n\}$ Cauchy, so $\{u_n\}$ converging.

$\boxed{\text{Yes}}$

5k) $\boxed{\text{Yes.}}$

$$u_1 = (0, \ldots, 0)$$
$$u_2 = (1, 0, \ldots, 0)$$
$$u_3 = (1 - 1/2, 0, \ldots, 0)$$
$$u_4 = (1 - 1/2 + 1/3, 0, \ldots, 0)$$
$$u_5 = (1 - 1/2 + 1/3 - 1/4, 0, \ldots, 0)$$
$$u_6 = (1 - 1/2 + 1/3 - 1/4 + 1/5, 0, \ldots, 0)$$
$$\vdots$$

Converges.

5l) Let $U = \mathbb{Q}^n$. Then bd U is $\mathbb{R}^n$ which is open. $\boxed{\text{Yes.}}$

6) K seq. compact, so f attains its minimum for some $x_0 \in K$.

Let $r = f(x_0)/2$. Then $r > 0$, and $f(x) > r$.

(If K is empty, any $r > 0$ will do.)

7a) cos is Lipschitz mapping, so $\boxed{\text{Yes}}$

7b) x^2 is Lipschitz on $[-2, 2]$, so $\boxed{\text{Yes}}$ at least for some ε, no solution for all t however.

7c) $\boxed{\text{No.}}$ Example from class.

$$y(t) = \begin{cases} 0 & t < t_0 \\ \frac{(t-t_0)^2}{4} & t \geq t_0 \end{cases}$$ is a solution for every $t_0 \geq 0$.

8)

Assume V is open let $u \in f^{-1}(V)$. We need to show that there is $\delta > 0$ such that $B_\delta(u) \subset f^{-1}(V)$. Let $v = f(u)$.

Because V is open, there is $\varepsilon > 0$ such that $B_\varepsilon(v) \subset V$.

By 1) there is $\delta > 0$ such that

for all $x \in X$ $\operatorname{dist}(u, x) < \delta \Rightarrow \operatorname{dist}(f(u), f(x)) < \varepsilon$.

If $x \in B_\delta(u)$, then $\operatorname{dist}(u, x) < \delta$, so $\operatorname{dist}(f(u), f(x)) < \varepsilon$,

so $\operatorname{dist}(v, f(x)) < \varepsilon$, so $f(x) \in B_\varepsilon(v) \subset V$.

Hence, $B_\delta(u) \subset f^{-1}(V)$.

9). We know that $\operatorname{int} U \cup \operatorname{bd} U \cup \operatorname{ext} U = \mathbb{R}^n$

that $\operatorname{int} U$, $\operatorname{ext} U$ open,

and that $\operatorname{int} U \cap \operatorname{bd} U = \operatorname{ext} U \cap \operatorname{bd} U$

$= \operatorname{int} U \cap \operatorname{ext} U = \emptyset$.

Assume that $\operatorname{bd} U = \emptyset$, and $\operatorname{int} U \neq \mathbb{R}^n$ and $\operatorname{int} U \neq \emptyset$.

Then $\operatorname{int} U$ and $\operatorname{ext} U$ separate $\mathbb{R}^n$, so by Definition $\mathbb{R}^n$ is not connected. A contradiction, so $\operatorname{int} U = \mathbb{R}^n$ or $\operatorname{int} U = \emptyset$. In the first case, since $U \supset \operatorname{int} U = \mathbb{R}^n$, $U = \mathbb{R}^n$. In the second case, $\operatorname{ext} U = \mathbb{R}^n$, and

$\mathbb{R}^n \setminus U \supset \underset{\overset{\shortparallel}{\mathbb{R}^n}}{\operatorname{ext} U}$, $\mathbb{R}^n \setminus U = \mathbb{R}^n$, so U empty.