Topics for the Final Exam, Math 411, Spring 2012

May 12, 2012

Here, "Lecture notes" refers to the notes on `http://math.berkeley.edu/~matthias/math411/`, in particular, the notes "What is integration" for April 9th and "Characterization of Riemann integrable functions" for April 26th. Other references are with respect to the textbook "Advanced Calculus" (Second edition) by Patrick M. Fitzpatrick.

Good practice problems are Mike Boyle's Exam from Fall 2011, Hisham Sati's Exam from Spring 2011 (Problems 5 and 6), and the practice problems for the lecture on May 9th (those and the other two exams can be found at `http://math.berkeley.edu/~matthias/math411/` under the "Syllabus" section, some solutions are available at the test bank, see section "Other links").

- Definition of Partial and Directional Derivative (Section 13.3, in particular Theorem 13.16)
- Inverse and Implicit Function Theorem (Theorem 16.12, Theorem 17.6)
- Definition of Riemann-integral through lower and upper Darboux sums (Section 18.1, also see lecture notes for April 9th)
- Theorem: A function $f : \mathbb{R}^n \to \mathbb{R}$ is Riemann-integrable if and only if it is bounded, has compact support, and the set of discontinuities has Lebesgue-measure zero. (See lecture notes April 26th for the statement of the theorem, recall that compact support means that the set of points where f is non-zero is bounded.)
- Examples of Riemann-integrable and non-Riemann-integrable functions such as the Dirichlet function/characteristic function of $\mathbb{Q} \cap [0, 1]$, the function $f(p/q) = 1/q$ for $0 < p < q$ coprime, and $f(x) = 0$ otherwise.

- Using the definition of the Riemann integral through lower and upper Darboux sums of standard partitions, the properties 1-6 in the lecture notes for April 9th follow. I proved this in lecture. The exam will ask for a proof of a similar property. An example of a proof which illustrates a lot of the ideas but is much more technical and complicated than what is asked in the exam is in the lecture notes for April 26th.

- The sequence of grad, curl, and div:

  - Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. Then curl $\vec{F} = 0$ if and only if there is a smooth scalar field $g : \mathbb{R} \to \mathbb{R}$ with grad $g = \vec{F}$. The scalar field is unique up to a constant.
  - Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. Then div $\vec{F} = 0$ if and only if there is a smooth vector field $\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$ with curl $\vec{G} = \vec{F}$.

- Stoke's Theorem: For a smooth vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$, we have

  $$\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl } \vec{F} \cdot d\vec{A}$$

  where A is a surface with boundary being the loop C. (See Theorem 20.31)