No books, notes or calculators are allowed. To receive full credit you must show all of your work. You have 15 minutes to complete the quiz.

Name (Last, First) and UID: ______________________

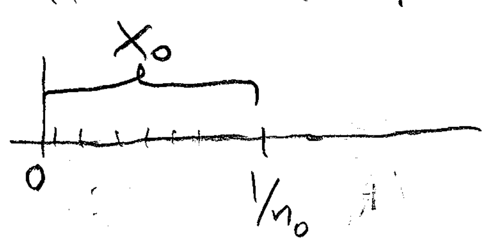
1. Let $X = \{\frac{1}{n} \text{ where n in } \mathbb{N}\} \subset \mathbb{R}$. Let $\chi_X : \mathbb{R} \to \mathbb{R}$ be the characteristic function on X defined by $\chi_X(x) = 1$ if x in X and $\chi_X(x) = 0$ otherwise. What is the set of x such that χ_X is not continuous at x? What is the Lebesgue measure of that set and why? Conclude using the theorem characterizing Riemann-integrable functions whether χ_X is Riemann-integrable.

2. Let X and χ_X be as above. Let $\epsilon > 0$. Let n_0 be an integer such that $\frac{1}{n_0} < \frac{\epsilon}{4}$ and k be an integer such that $2^{-k} < \frac{1}{n_0}$ and $n_0 2^{-k} < \frac{\epsilon}{2}$. Let $X_0 = \{x \in X | x < \frac{1}{n_0}\}$ and $X_1 = \{x \in X | x \geq \frac{1}{n_0}\}$. Give a bound for $U(\chi_{X_0}, P_k), U(\chi_{X_1}, P_k)$, and $U(\chi_X, P_k)$ where P_k is the standard partition of $\mathbb{R}$ into intervals of length 2^{-k} with one interval starting at 0. Without using the above mentioned theorem, conclude whether χ_X is Riemann-integrable.
Hint: What is the size of X_1?

1. χ_X not continuous on $\{0\} \cup X$.

$\{0\} \cup X$ is countable, so Lebesgue measure 0, so

χ_X bounded, compact support, so χ_X Riemann-integrable.

2. 

$X_0 \subset [0, 1/n_0]$ and $[0, 1/n_0]$ covered by at most $1 + 1/n_0 / 2^{-k}$ rectangles which have vol $2^{-k}(1 + 1/n_0 / 2^{-k}) = 2^{-k} + 1/n_0 < \varepsilon/2$

$U(\chi_{X_0}, P_k) < \varepsilon/2$

X_1 has n_0 pts, so covered by at most n_0 rectangles, total vol $< n_0 \cdot 2^{-k} < \varepsilon/2$

$U(\chi_{X_1}, P_k) < \varepsilon/2$

$X = X_0 \cup X_1$, so $U(\chi_X, P_k) \leq U(\chi_{X_0}, P_k) + U(\chi_{X_1}, P_k) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$.

$L(\chi_X, P_k) = 0$, so $\lim L(\chi_X, P_k) = \lim U(\chi_X, P_k) = 0$.