Solutions to Problem 5 from Section 11.4

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Problem 0.1. *Let K be a sequentially compact subset of $\mathbb{R}^n$ and suppose that $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ that contains K. Prove that there is some positive number r such that for any point $\mathbf{u}$ in K, $\mathcal{B}_r(\mathbf{u}) \subseteq \mathcal{O}$.*

Remark 0.2. Notice the order of the quantifiers, "there is ... such that for any ...". If you switch them to "for any point $\mathbf{u}$ in K, there is some positive number r such that $\mathcal{B}_r(\mathbf{u}) \subseteq \mathcal{O}$", the problem becomes trivial because every point u in K is in $\mathcal{O}$ and, hence, the property follows by the definition of $\mathcal{O}$ being open (Definition in 10.3).

1 Solution 1

1.1 Solution Idea

Notice that $\mathcal{O}$ is open, so for every $\mathbf{u}$ there is a positive number r with $\mathcal{B}_r(\mathbf{u}) \subseteq \mathcal{O}$. In other words, this r exists pointwise, but we ask for a uniform r, i.e., one that works on all of K. If there is no such positive r, then there must be a sequence of points $\{\mathbf{u}_k\}$ in K and radii $\{r_k\}$ such that the sequence of balls $\mathcal{B}_{r_k}(\mathbf{u}_k)$ is not contained in $\mathcal{O}$ and r_k converging to zero. But K is sequentially compact, so $\{\mathbf{u}_k\}$ has a subsequence convering to a point $\mathbf{u}$ in K. At that point, no open ball $\mathcal{B}_r$ will be contained in $\mathcal{O}$. A contradiction to $\mathcal{O}$ being open.

1.2 Solution

Proof of Problem 0.1. If K is empty, every r suffices. So assume K is non-empty.

Notice that, by the definition of $\mathcal{O}$ being open, for every $\mathbf{u}$ in K, there is such a positive number r. If no such number r working for all points in K simultaneously exists, then there must be a sequence of $\{r_k\}$ converging to zero such that for each r_k there is a point $\mathbf{u}_k$ in K such that the ball $\mathcal{B}_{r_k}(\mathbf{u}_k)$ is not contained in $\mathcal{O}$. You can pick for each k such a point and thus get a sequence $\{\mathbf{u}_k\}$. Furthermore, there is a point v_k in $\mathcal{B}_{r_k}(\mathbf{u}_k)$ not in $\mathcal{O}$, i.e., there is a sequence of points $\{v_k\}$ not in $\mathcal{O}$ such that $d(\mathbf{u}_k, \mathbf{v}_k) < r_k$.

Because K is sequentially compact, there is a subsequence $\{\mathbf{u}_{k_i}\}$ of $\{\mathbf{u}_k\}$ that is converging to a point $\mathbf{u}$ in K. At $\mathbf{u}$, there is a positive r such that $\mathcal{B}_r(\mathbf{u}) \subseteq \mathcal{O}$ because $\mathcal{O}$ is open. Pick the corresponding subsequence $\{\mathbf{v}_{k_i}\}$ and notice that as $d(\mathbf{u}_k, \mathbf{v}_k) < r_k \to 0$, the subsequence converges to $\mathbf{u}$. Hence, there is some $\mathbf{v}_{k_i}$ with $d(\mathbf{u}, \mathbf{v}_{k_i}) < r$, so it is in $\mathcal{B}_r(\mathbf{u}) \subseteq \mathcal{O}$. But that is a contradiction because we picked v_k not to be in $\mathcal{O}$. □

2 Solution 2

2.1 Solution Idea

Consider the function $f : K \to \mathbb{R}$ that assigns to a point $\mathbf{u}$ in K the largest r such that $\mathcal{B}_r(\mathbf{u}) \subseteq \mathcal{O}$. Intuitively speaking, if $\mathcal{O}$ were a "sane" open set, $f(\mathbf{u})$ is the distance of $\mathbf{u}$ to the boundary bd$\mathcal{O}$ of $\mathcal{O}$. The problem now translates to: There is some positive number r such that $f(\mathbf{u}) \geq r$ for all $\mathbf{u}$ in K.

f is a continuous function because if you move $\mathbf{u}$ a bit, the distance to the boundary cannot change by too much either. Even stronger, f is a Lipschitz mapping (Definition in 12.2) with Lipschitz constant 1 because if you move $\mathbf{u}$ by a little amount Δ, then the distance to the boundary cannot change by more than $|\Delta|$. Because f is continuous and K is compact, the image $f(K)$ is also compact, and being a compact subset of $\mathbb{R}$, it attains its minimum. Because $\mathcal{O}$ is open, f is nowhere zero. Hence, the minimum of f cannot be zero, but must be a positive number r which will fulfill the conditions of the problem.

2.2 Solution

Definition 2.1. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ such that $\mathcal{O} \neq \mathbb{R}^n$. Let $f : \mathcal{O} \to \mathbb{R}$ be defined by sending a point $\mathbf{u}$ in $\mathcal{O}$ to the maximal r such that $\mathcal{B}_r(\mathbf{u}) \subseteq \mathcal{O}$.*

Lemma 2.2. *f is well-defined and everywhere positive.*

Proof. Fix $\mathbf{u}$ in $\mathcal{O}$. We need to show that there is such a maximal r. Consider the set $M \subseteq \mathbb{R}$ of all r such that $\mathcal{B}_r(\mathbf{u}) \subseteq \mathcal{O}$.
First, notice that $\mathcal{O}$ is open, so there is some positive r with $\mathcal{B}_r(\mathbf{u}) \subseteq \mathcal{O}$. Hence, M contains at least one positive number and is non-empty.
Notice that $\mathcal{O} \neq \mathbb{R}^n$, so there is a point $\mathbf{v}$ not in $\mathcal{O}$. For any $r > d(u, v)$, we have $\mathcal{B}_r(\mathbf{u}) \nsubseteq \mathcal{O}$. Hence, M is bounded by $d(\mathbf{u}, \mathbf{v})$.
Furthermore, an increasing sequence $\{r_k\}$ of numbers in M yields a sequence of balls $\{\mathcal{B}_{r_k}\}$ and, because each is contained in $\mathcal{O}$, so is there union which is a ball $\mathcal{B}_r$ with r being the limit of $\{r_k\}$.
Hence, M contains its maximum. And M contains a positive number, so $f(\mathbf{u}) > 0$. □

Lemma 2.3. *f is a Lipschitz mapping with Lipschitz constant 1 (Definition in 12.2).*

Proof. Let $\mathbf{u}, \mathbf{v}$ be points in $\mathcal{O}$. Let $\mathcal{B}_r(\mathbf{u})$ with $r = f(\mathbf{u})$ be the largest ball around $\mathbf{u}$.
If $\mathbf{v}$ is in $\mathcal{B}_r(\mathbf{u})$, then $d(\mathbf{u}, \mathbf{v}) < \mathbf{v}$ and $\mathcal{B}_{r-d(\mathbf{u},\mathbf{v})}(\mathbf{v})$ is a subset of $\mathcal{B}_r(\mathbf{u})$ by the triangle inequality, and hence a subset of $\mathcal{O}$. So, $f(\mathbf{v}) \geq r - d(\mathbf{u}, \mathbf{v}) = f(\mathbf{u}) - d(\mathbf{u}, \mathbf{v})$, so

$$f(\mathbf{u}) - f(\mathbf{v}) \leq d(\mathbf{u}, \mathbf{v})$$

.
If $\mathbf{v}$ is not in $\mathcal{B}_r(\mathbf{u})$, then $d(\mathbf{u}, \mathbf{v}) \geq r = f(\mathbf{u})$. Since $f(\mathbf{v})$ is positive, we again have

$$f(\mathbf{u}) - f(\mathbf{v}) \leq d(\mathbf{u}, \mathbf{v}).$$

Interchanging $\mathbf{u}$ and $\mathbf{v}$ in the argument, we get $-(f(\mathbf{u}) - f(\mathbf{v})) \leq d(\mathbf{u}, \mathbf{v})$, so

$$|f(\mathbf{u}) - f(\mathbf{v})| \leq d(\mathbf{u}, \mathbf{v}).$$

□

Lemma 2.4. *If f is a Lipschitz mapping with Lipschitz constant C, then f is continuous.*

Proof. Pick a point $\mathbf{u}$ and a real number $\epsilon > 0$. Now, pick $\delta = \epsilon/C$. For any point $\mathbf{v}$ with $d(\mathbf{u}, \mathbf{v}) < \delta$, we have $|f(\mathbf{v}) - f(\mathbf{u})| \leq Cd(\mathbf{u}, \mathbf{v}) < C\delta = C\epsilon/C = \epsilon$. □

Lemma 2.5. *If $M \subseteq \mathbb{R}$ is non-empty, sequentially compact and only contains positive numbers, then it has a minimum r in M and $r > 0$.*

Proof. Sequentially compact (Theorem 11.18) implies bounded and closed. Pick a sequence converging to the infimum of M. Because M is closed, the limit is contained in M and will be the minimum r of M. Because r is in M, it is a positive number. □

Proof of Problem 0.1. If $\mathcal{O} = \mathbb{R}^n$ or K is empty, any positive r will suffice. So let's assume $\mathcal{O} \neq \mathbb{R}^n$ and $K \neq \emptyset$.

By Lemma 2.3 and 2.4, f is continuous. The image $f(K)$ is sequentially compact because K is sequentially compact (Theorem 11.20). By Lemma 2.2, $f(K)$ is non-empty and contains only positive numbers. By Lemma 2.5, M has a minimum r. So for every $\mathbf{u}$ in K, $f(\mathbf{u}) \geq r$, and hence the ball $B_r(\mathbf{u})$ is contained in $\mathcal{O}$. □