Cheat sheet for continuous functions

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1 Sequentially compact, pathwise-connected, connected are mapped forward, open and closed are mapped backward

Theorem 1.1. *Let K be sequentially compact, pathwise-connected, respectively, connected. Let $f : K \to Y$ be a continuous function. Then $f(K)$ is sequentially compact, pathwise-connected, respectively, connected.*

Theorem 1.2. *Let $f : X \to Y$ be a continuous function. Let A be an open, respectively, closed subset of Y. Then $f^{-1}(A)$ is open, respectively, closed in X. In particular, if X is an open subset of $\mathbb{R}^n$, then $f^{-1}(A)$ is open, and if X is a closed subset of $\mathbb{R}^n$, then $f^{-1}(A)$ is closed.*

These theorems are valid for any topological or metric space X and Y.

2 Convex implies pathwise-connected implies connected

Theorem 2.1. *Let A be a subset of $\mathbb{R}^n$. If A is convex, then A is also pathwise-connected. If A is pathwise-connected, then A is also connected.*

The first statement of the theorem only makes sense if A is an affine space or vector space. The second statement of the theorem is valid for all topological or metric spaces.

Theorem 2.2. *Let A be an open subset of $\mathbb{R}^n$. Then, A is pathwise-connected if and only if A is connected.*

This theorem only works for topological spaces locally pathwise-connected.

3 Lipschitz implies uniform continuous implies continuous, the last two are equivalent on sequentially compact domain

Theorem 3.1. *If $f : X \to Y$ is a Lipschitz mapping with Lipschitz constant C, then f is uniform continuous. If f is uniform continuous, then f is continuous.*

Theorem 3.2. *If X is sequentially compact and $f : X \to Y$ is continuous, then f is uniform continuous.*

These theorems work for all metric spaces.