SCALAR PRODUCTS, NORMS AND METRIC SPACES

1. Definitions

Below, "real vector space" means a vector space V whose field of scalars is $\mathbb{R}$, the real numbers. The main example for MATH 411 is $V = \mathbb{R}^n$. Also, keep in mind that "0" is a many splendored symbol, with meaning depending on context. It could for example mean the number zero, or the zero vector in a vector space.

Definition 1.1. A **scalar product** is a function which associates to each pair of vectors x, y from a real vector space V a real number, $< x, y >$, such that the following hold for all x, y, z in V and α in $\mathbb{R}$:

(1) $< x, x > \geq 0$, and $< x, x > = 0$ if and only if $x = 0$.
(2) $< x, y > = < y, x >$.
(3) $< x + y, z > = < x, z > + < y, z >$.
(4) $< \alpha x, y > = \alpha < x, y >$.

The **dot product** is defined for vectors in $\mathbb{R}^n$ as $x \cdot y = x_1 y_1 + \cdots + x_n y_n$. The dot product is an example of a scalar product (and this is the only scalar product we will need in MATH 411).

Definition 1.2. A **norm** on a real vector space V is a function which associates to every vector x in V a real number, $||x||$, such that the following hold for every x in V and every α in $\mathbb{R}$:

(1) $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$.
(2) $||\alpha x|| = |\alpha| ||x||$.
(3) (Triangle Inequality for norm) $||x + y|| \leq ||x|| + ||y||$.

The standard Euclidean norm on $\mathbb{R}^n$ is defined by $||x|| = \sqrt{x \cdot x}$. There are other useful norms, as we'll see.

Definition 1.3. A **metric space** is a set X together with a function d which associates to each pair of points x, y from X a real number, $d(x, y)$, such that the following hold for all x, y, z in X:

(1) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$.
(3) (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

Above, that function d is the distance function, also called the metric. An example of a distance function is the usual distance between vectors x, y in $\mathbb{R}^n$:

$$d(x, y) = ||x - y|| = \sqrt{< x - y, x - y >} .$$

2. Relations

What we see in the familiar examples above is perfectly general:

(1) Given a scalar product $< \cdot, \cdot >$ on a real vector space V, the definition $||x|| = \sqrt{< x, x >}$ define a norm on V.

(2) Given a norm $||\cdot||$ on V, the definition $d(x,y) = ||x-y||$ defines a distance function making V a metric space.

These two claims are checked simply by verifying that the defining properties hold. For example, to show property (1) of a metric space is true for the function $d(x,y) = ||x-y||$, you note that it easily follows from property (1) for the norm. All the other properties are likewise easy to check, except for checking that the triangle inequality for the norm holds when $||x||$ is defined by $\sqrt{< x,x >}$. For this, you prove the Cauchy Schwarz Inequality first, then make the argument. The proof of the Cauchy Schwarz Inequality only uses the properties listed in the definition of the scalar product.

3. Important norms on $\mathbb{R}^n$

To consider more than one norm, we use a little more notation. For x in $\mathbb{R}^n$, define the following three norms:

$$\begin{aligned} ||x||_1 &= |x_1| + \cdots + |x_n| \\ ||x||_2 &= \sqrt{(x_1)^2 + \cdots + (x_n)^2} \\ ||x||_\infty &= \max\{|x_1|, |x_2|, \ldots, |x_n|\} \ . \end{aligned}$$

The norm $||\cdot||_2$ is also called the standard norm, or the Euclidean norm. The norm $||\cdot||_{\infty|}$ is also called the supremum norm, or the sup norm.

Associated to each of these norms is the distance function it defines. Let us call these respectively d_1, d_2, d_∞. The distance d_2 is the Euclidean distance – it is the standard idea of distance on $\mathbb{R}^n$, known to Euclid (when $n \leq 3$).

Exercise. Draw the "unit circle" for each of these three metrics for the case $\mathbb{R}^2$, where "unit circle" means the set of points which are distance 1 from the origin.

4. Convergence

For a sequence $x_1, x_2, \ldots$ of points in a metric space X, convergence of the sequence to a point x in X is denoted $\lim_{k\to\infty} x_k = x$. We define this to be true if and only if

$$\lim_{k\to\infty} \mathrm{dist}(x_k, x) = 0 \ .$$

Going back to MATH 410: this last condition means that for any $\epsilon > 0$, there is a number M such that

$$k > M \implies \mathrm{dist}(x_k, x) < \epsilon \ .$$

We will see below that the distances d_1, d_2, d_∞ define the same notion of convergence on $\mathbb{R}^n$: if $\lim_{k\to\infty} \mathrm{dist}(x_k, x) = 0$ if d is any one of these three, then $\lim_{k\to\infty} \mathrm{dist}(x_k, x) = 0$ if it is one of the others as well.

5. Some inequalities

For real numbers $x_1, \ldots, x_n$: considering cross terms we see

$$|x_1|^2 + \cdots + |x_n|^2 \ \leq \ (|x_1| + \cdots + |x_n|)^2$$

and therefore, taking square roots,

$$\sqrt{|x_1|^2 + \cdots + |x_n|^2} \ \leq \ |x_1| + \cdots + |x_n| \ .$$

We can add couple of easily verified inequalities to get:

$$\max_i |x_i| \;\le\; \sqrt{|x_1|^2 + \cdots + |x_n|^2} \;\le\; |x_1| + \cdots + |x_n| \;\le\; n \max_i |x_i| \ .$$

In the notation of norms, we can write the last line as

$$||x||_\infty \;\le\; ||x||_2 \;\le\; ||x||_1 \;\le\; n||x||_\infty \ .$$

We see that given any two of these norms, $||\cdot||_i$ and $||\cdot||_j$ say, that there is a positive constant C such that $||x||_i \le C||x||_j$ for all x. Consequently, whenever

$$\lim_k d_j(x_k, x) = 0$$

it must also be true that

$$\lim_k d_i(x_k, x) = 0$$

which means that convergence of a sequence to x in the d_j metric implies convergence of the sequence to x in the d_i metric.

So! If we want to check convergence of a sequence of points in $\mathbb{R}^n$, we can use any of the three criteria, at our convenience

A closely related equivalent criterion for convergence of a sequence x_k to a point x in $\mathbb{R}^n$ is **componentwise convergence**, as discussed in the text of Fitzpatrick.

6. Some remarks

Definition 6.1. Two norms (call them $||\cdot||_a$ and $||\cdot||_b$) on a real vector space V are **equivalent** if there exist positive numbers C_1, C_2 such that for all x in V,

$$\begin{aligned} ||x||_a &\le C_1||x||_b \ , \quad \text{and} \\ ||x||_b &\le C_2||x||_a \ . \end{aligned}$$

Here is a fact. Any two norms on $\mathbb{R}^n$ are equivalent! So, all norms on $\mathbb{R}^n$ determine the same notion of convergence – this is not a special property of the particular three norms we looked at above. Proving that would be an interesting exercise within your powers after we finish the early part of the course on metric space topology.

The definitions of scalar product and norm generalize to infinite dimensional vector spaces, and especially to spaces of functions. That is one reason for spelling out the chain of logic here.

There are many metrics which do not arise from norms. For example, if we take a subset of a metric space, with the same definition for distance, it is again a metric space. For example, the circle in $\mathbb{R}^2$ with the standard metric is a metric space.

Also, in contrast to the situation for norms, there are (vastly) many metrics on $\mathbb{R}^n$ which are not equivalent.