Solution to an earlier Midterm problem

March 2, 2012

Problem 0.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the partial derivatives exists everywhere and are bounded, i.e., there is an $M > 0$ such that $\left|\frac{\partial f}{\partial x}(\mathbf{u})\right| < M$ and $\left|\frac{\partial f}{\partial y}(\mathbf{u})\right| < M$ for all $\mathbf{u}$ in $\mathbb{R}^2$. Show that f is continuous.*

Remark 0.2. We will prove the even stronger result that f is a Lipschitz mapping with Lipschitz constant $2M$, i.e., $|f(\mathbf{u}) - f(\mathbf{v})| < 2Md(\mathbf{u}, \mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$.

Proof. Consider fixing y and regarding f as a function in x only, i.e., for given y, define $g_y : \mathbb{R} \to \mathbb{R}$ by $g_y(x) = f(x, y)$. Then, the derivative of g_y exists because the partial derivatives of f exists and $g_y'(x) = \frac{\partial f}{\partial x}(x, y)$, hence, $|g_y'(x)| < M$. Since, g_y is a function from $\mathbb{R}$ to $\mathbb{R}$ and is differentiable, we know that the Mean Value Theorem holds and that g_y is continuous. The Mean Value Theorem says that, for every x and x_0 in $\mathbb{R}$, there is a $\tilde{x}$ between x and x_0 such that $\frac{g_y(x_0) - g_y(x)}{x_0 - x} = g_y'(\tilde{x})$. We get

$$g_y(x_0) - g_y(x) = g_y'(\tilde{x})(x_0 - x)$$

and hence

$$|g_y(x_0) - g_y(x)| < M|x_0 - x|.$$

Translating it back to f, it means that for all x_0, x, and y in $\mathbb{R}$:

$$|f(x_0, y) - f(x, y)| < M|x_0 - x|. \tag{1}$$

Similarly, we can fix x and regard f as a function in y, and we obtain:

$$|f(x, y_0) - f(x, y)| < M|y_0 - y|.$$

Replacing x by x_0 we get:

$$|f(x_0, y_0) - f(x_0, y)| < M|y_0 - y| \tag{2}$$

Applying the triangle inequality to equation 1 and 2, we obtain

$$|f(x_0, y_0) - f(x, y)| < M(|x_0 - x| + |y_0 - y|).$$

Letting $\mathbf{u} = (x, y)$ and $\mathbf{v} = (x_0, y_0)$, we see that $|x_0 - x| < d(\mathbf{u}, \mathbf{v})$ and $|y_0 - y| < d(\mathbf{u}, \mathbf{v})$, so

$$|f(\mathbf{u}) - f(\mathbf{v})| < 2Md(\mathbf{u}, \mathbf{v}).$$

This holds for all x, x_0, y, y_0 in $\mathbb{R}$, hence for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$. □