1a) Note: $f: \mathbb{R} \to \mathbb{R}$ so differentiable implies continuous.

Yes

1b) Yes

1c) Can't tell

1d) No, otherwise Hessian matrix is symmetric.

2) $\det Df(1) = 1 \cdot 1 - 2 \cdot 2 = -3 \neq 0$, so by Inverse Function Thm, f locally invertible. Yes.

3) $\det \nabla^2 f(0) = 1 \cdot 3 - 2 \cdot 2 = -1 < 0$, so no local extremum at 0. But potentially a local extremum somewhere else.

4) Df constant, so f is linear $f(x) = Ax$ with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \det A = -1 \neq 0,$$

so A invertible and f's inverse is given by

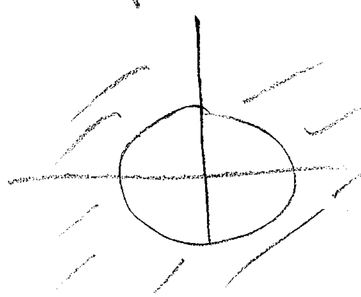
$$g(y) = A^{-1} y$$

5) $f: \mathbb{R}^2 \to \mathbb{R}^3$. Note that $3 > 2$, so not surjective and hence not invertible. Submatrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ of Df is invertible because $\det = 1 \neq 0$, so f is injective because $\pi \circ f$ with $\pi(x, y, z) = (x, y)$ is injective.

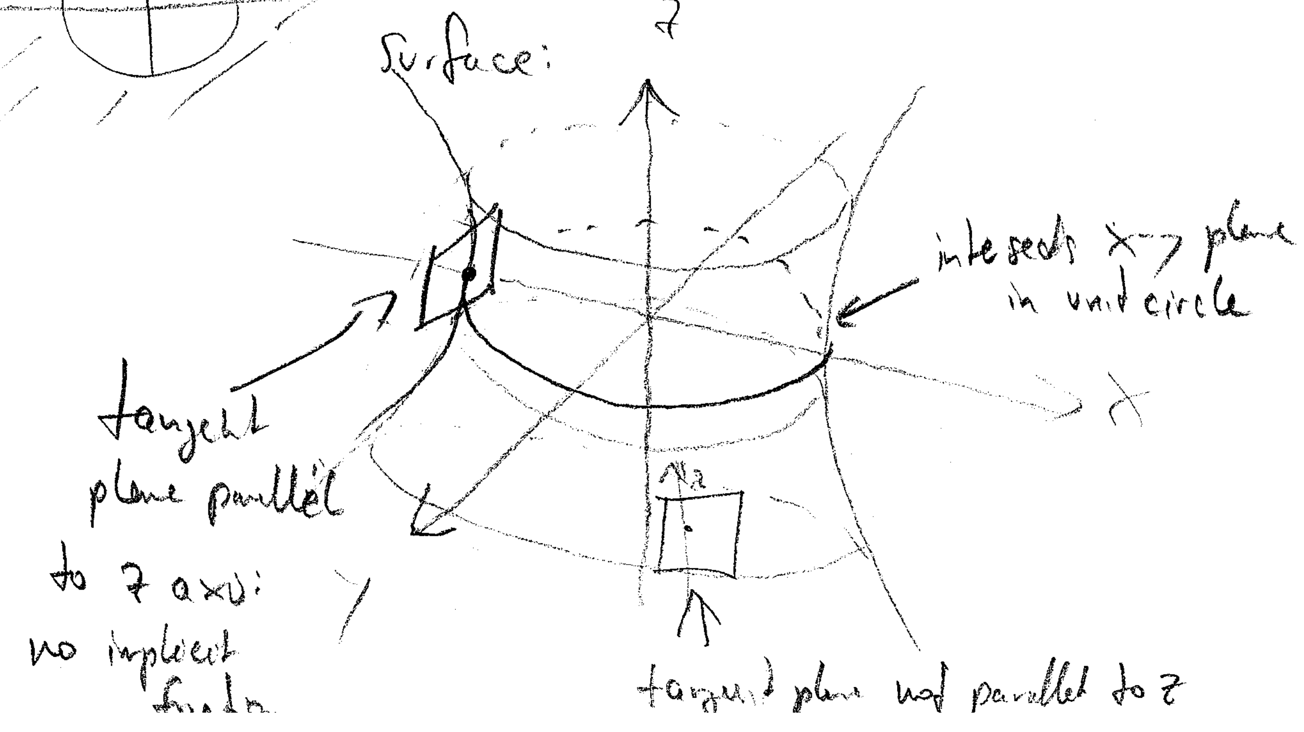
7) $\nabla g = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$ g everywhere smooth and

$\nabla g = 0$ only at $(0,0,0)$ but $(0,0,0) \notin g^{-1}(0)$, so $g^{-1}(0)$ is a surface without singularity.

$x^2+y^2 = z^2+1 \geq 1$ has no solution if $x^2+y^2<1$, so implicit function f does not exists for those points. For $x^2+y^2=1$, $z=0$, and $\frac{\partial g}{\partial z} = 0$, so implicit function would have infinite derivative, not differentiable. For $x^2+y^2>1$, $\frac{\partial g}{\partial z} \neq 0$. So f exists on



$x^2+y^2>1$.



6)

$\varepsilon = 0.001$

$$\frac{\partial f}{\partial y}(0,0) \approx \frac{f(0,\varepsilon) - f(0,0)}{\varepsilon}$$

$$\frac{\partial f}{\partial y}(\varepsilon,0) \approx \frac{f(\varepsilon,\varepsilon) - f(\varepsilon,0)}{\varepsilon}$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \approx \frac{\frac{\partial f}{\partial y}(\varepsilon,0) - \frac{\partial f}{\partial y}(0,0)}{\varepsilon}$$

$$= \frac{f(\varepsilon,\varepsilon) - f(\varepsilon,0) - f(0,\varepsilon) - f(0,0)}{\varepsilon^2}$$

$$= \frac{1.002 - 1.001 - 0.999 + 1.003}{0.001^2}$$

$$= \frac{0.005}{0.001 \cdot 0.001} = 5000$$