This exam takes about 1.5 to twice the time of the real exam.

1. Answer each of the following questions with yes/no/can't tell. No justification necessary.
   (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that all partial derivatives exist everywhere. Is f continuous?
   (b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that all second order partial derivatives exist everywhere. Is $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$?

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuously differentiable function such that
$$Df(0) = \begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & 6 \\ 1 & 1 & 2 \end{pmatrix}.$$
Is f locally invertible near 0?

3. Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be a continuously differentiable function such that $Df(0) = \begin{pmatrix} 1 & 3 & 2 & 4 \\ -1 & 1 & 2 & 0 \\ 1 & 2 & 4 & 5 \end{pmatrix}$.
Is f locally invertible at 0? Is f (surjective) onto a neighborhood around $f(0)$? Is f (injective) one-to-one on a neighborhood around 0?

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a twice continuously differentiable function. Write an expression E in $\epsilon, f(x,y), f(x+\epsilon, y), f(x, y+\epsilon), f(x+\epsilon, y+\epsilon)$ such that
$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \lim_{\epsilon \to 0} E.$$

5. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by $g(x,y,z) = \left(\sqrt{x^2+y^2} - 2\right)^2 + z^2 - 1$. Is the subspace $g^{-1}(0)$ of points (x,y,z) with $g(x,y,z) = 0$ a surface without singularities? If no, give a singularity. If yes, prove there are no singularities. For which (x,y) does there locally exists a continuously-differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ that fulfills $g(x,y,f(x,y)) = 0$?

6. Write a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(1,1) = 1, \nabla f(1,1) = (0,1), \nabla^2 f(1,1) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$?

7. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a continuously differentiable function with $f(0) = 0, Df(0) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 8 \end{pmatrix}$.
Is there an implicitly defined function g near 0 such that $f(x, g(x)) = 0$? What is $Dg(0)$?

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a twice continuously differentiable function with $\nabla^2 f(x,y) = \begin{pmatrix} x & 1 \\ 1 & 2 \end{pmatrix}$ and $\nabla f(0,0) = (1,1)$. Compute $\nabla f(1,0)$.

9. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable functions with $f(0) = g(0) = 0$ and $Df(0) = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ and $D(g \circ f)(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. What is $Dg(0)$?

10. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function with $\det Df(0) \neq 0$. Is f globally invertible, i.e., does there exists $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that $g \circ f = f \circ g = \mathrm{id}$, i.e., $g(f(x)) = f(g(x)) = x$ for all x? If no, give a counterexample. If yes, give a precise argument.