To receive full credit, you must show all of your work. No books, notes or electronic equipment are allowed. You have 50 minutes to complete the exam.
Please write your name (last, first) and UID at the top right of each page.

Good luck!

20 P. 1. Answer each of the following questions with yes/no/can't tell. No justification necessary.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that all partial derivatives exist everywhere. Is f continuous?

(b) Let $\mathcal{O}$ be a non-empty open set in $\mathbb{R}^3$. Let $f : \mathcal{O} \to \mathbb{R}$ be a function such that all partial derivatives exists everywhere and are continuous. Is f continuous?

(c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Do all second order partial derivatives exist?

(d) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the Hessian matrix $\nabla^2 f(0)$ of second order partial derivatives is $\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$. Is f twice continuously differentiable at 0?

10 P. 2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable function such that $Df(1) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Is f invertible near 1?

10 P. 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a twice continuously differentiable function with $\nabla^2 f(0) = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $\nabla f(0) = 0$. Does f have a local extremum?

20 P. 4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function such that Df is constant to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Is f globally invertible? If no, give a counterexample. If yes, give a precise argument.
(Globally invertible means that there exists $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that $g \circ f = f \circ g = \mathrm{id}$, i.e., $g(f(x)) = f(g(x)) = x$ for all x.)

25 P. 5. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a continuously differentiable function such that $Df(1) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$. Is f locally invertible at 1? Is f (surjective) onto a neighborhood around $f(1)$? Is f one-to-one (injective) on a neighborhood around 1?

20 P. 6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a twice continuously differentiable function such that $f(0.000, 0.000) = 1.003, f(0.001, 0.000) = 1.001, f(0.000, 0.001) = 0.999, f(1.001, 1.001) = 1.002$. Approximate $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$.

35 P. 7. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by $g(x, y, z) = x^2 + y^2 - z^2 - 1$. Is the subspace $g^{-1}(0)$ of points (x, y, z) with $g(x, y, z) = 0$ a surface without singularities? If no, give a singularity. If yes, prove there are no singularities. For which (x, y) does there locally exists a continuously-differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ that fulfills $g(x, y, f(x, y)) = 0$?

Total: 140