Recap: Topology of $\mathbb{R}^n$

open U $\xleftrightarrow[\text{complement}]{}$ closed

Def Open:

Def: every point $u \in U$ has a ball $B_r(u) \subset U$ **Def:** U is union of balls $B_r(u) \subset U$

Def: Converging Sequence $\{u_n\} \subset \mathbb{R}^n \setminus U$ has limit $u \in \mathbb{R}^n \setminus U$

Def: $\text{int}\, U = U$

Basic properties

Union of open sets is open

Finite intersection of open sets is open $\Longleftrightarrow$ analogous for closed sets

Decomposition: $\text{int}\, U \cup \text{bd}\, U \cup \text{ext}\, U = \mathbb{R}^n$

maximal open set $\subset U$ — everything else — maximal open set $\subset \mathbb{R}^n \setminus U$

Decision procedure: $u \in \mathbb{R}^n$ if, take smaller and smaller balls $B_r(u)$

if balls are eventually in U $\longrightarrow$ $u \in \text{int}\, U$

contained in $\mathbb{R}^n \setminus U$ $\longrightarrow$ $u \in \text{ext}\, U$

every ball intersects with U and $\mathbb{R}^n \setminus U$ $\longrightarrow$ $u \in \text{bd}\, U$

Examples: $B_r(U)$, inside, outside, bd is just the surface of ball,
int ext thickness zero, no measure,
but: necessary to glue together the open int and open ext

Crazy example: $\mathbb{Q}$ $\text{int}\, \mathbb{Q} = \text{ext}\, \mathbb{Q} = \emptyset$
$\text{bd}\, \mathbb{Q} = \mathbb{R}$

Exercises:
- open set such that $\text{bd}\, U$ not measure zero
- open U is union of countably many balls

Today:

Continuous Functions

Let $A \subset \mathbb{R}^n$ be open, $f: A \to \mathbb{R}^m$.

Equivalent Definitions of f being continuous:

1. For every converging sequence $\{u_k\} \subseteq A$ $\quad u_k \to u$

$$f(u_k) \text{ converges to } f(u)$$

2. For every point $u \in A$ and every $\varepsilon > 0$, there is an $\delta > 0$ such that

$$f(B_\delta(u)) \subset B_\varepsilon(f(u))$$

i.e., for every pt $v \in A$ with $\mathrm{dist}(u,v) < \delta$ we have $\mathrm{dist}(f(u), f(v)) < \varepsilon$

3. For every open set $V \subset \mathbb{R}^m$, $f^{-1}(V)$ is open.

(Equivalent: For every closed set $V \subset \mathbb{R}^m$, $f^{-1}(V)$ is closed ~~in A~~ if $A = \mathbb{R}^n$).

4. Write $F(u)$ as $(F_1(u), \ldots, F_m(u))$.

Each $F_i(u)$ is continuous.

if A open, $f^{-1}(V)$ is only closed in A.

Examples

constant

- Projection functions, sin, cos, exp, $\langle -,- \rangle$, $\|-\|$
- Sum, Product
- Quotient $\frac{f}{g}$ if $g(v) \neq 0$ for $v \in A$
- Composition $f \circ g$ defined by $(f \circ g)(x) = f(g(x))$

(– locally uniform convergence)

Examples: $F(u) = \frac{u}{\|u\|^2}$ continuous on $\mathbb{R}^n \setminus \{0\}$

$F(u) = \exp(\|u\|)$ on $\mathbb{R}^n$

Observation: (∞, c) open

$f^{-1}((\infty, c)) = \{u \in \mathbb{R}^n \mid f(u) < c\}$ open

$(\infty, c]$ closed

$f^{-1}((\infty, c]) = \{u \in \mathbb{R}^n \mid f(u) \leq c\}$ closed

$[c]$ closed

$f^{-1}(c) = \{u \in \mathbb{R}^n \mid f(u) = c\}$ closed

Exercise: $\mathcal{U}$ open $\Leftrightarrow$ $\exists f$ continuous s.t. $\mathcal{U} = f^{-1}((0, \infty))$

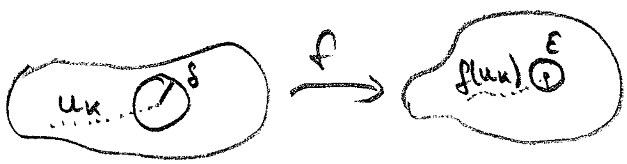
Thm: Above Defs are equivalent.

Proof (for $A = \mathbb{R}^4$)

$$\begin{tikzcd} \boxed{4} \arrow[r, Leftrightarrow] & \boxed{1} \arrow[d, Rightarrow] & \boxed{2} \arrow[l, Rightarrow] \\ & \boxed{3} \arrow[ur, Rightarrow] & \end{tikzcd}$$

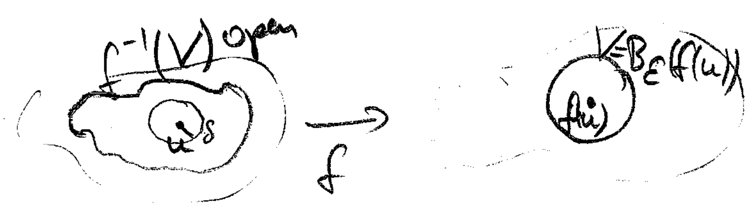
$\boxed{1} \Leftrightarrow \boxed{4}$ By convergence $\Leftrightarrow$ componentwise convergence Thm

$\boxed{2} \Rightarrow \boxed{1}$



Assume $\{u_K\} \subset A$ converging to u.
Pick $\varepsilon > 0$.
By $\boxed{2}$ there is $\delta > 0$ such that
$\mathrm{dist}(u,v) < \delta \Rightarrow \mathrm{dist}(f(u), f(v)) < \varepsilon$
By $\{u_K\}$ converging,
there is K such that
$k > K \Rightarrow \mathrm{dist}(u, u_K) < \delta$.
So, $\mathrm{dist}(\cancel{f}f(u), f(u_K)) < \varepsilon$.

$\boxed{3} \Rightarrow \boxed{2}$



Pick u, ε.
Let $V = B_\varepsilon(f(u))$.

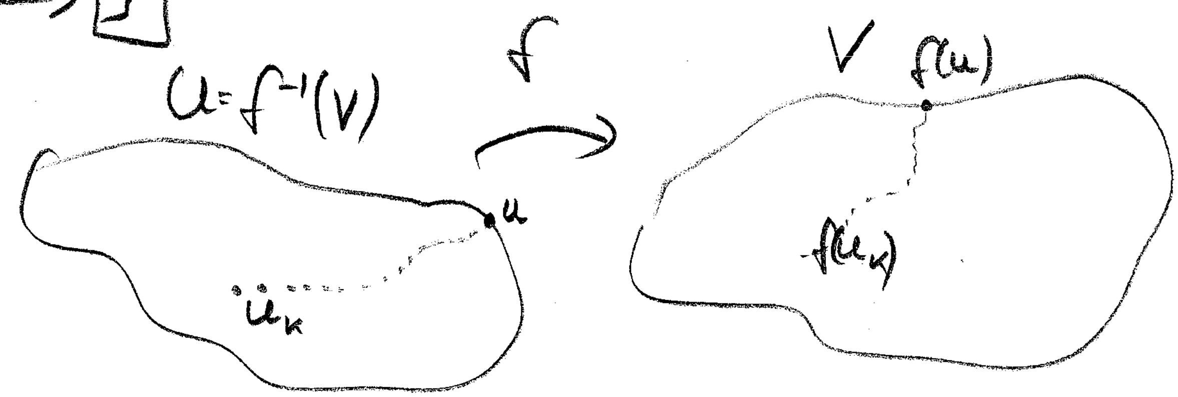
By $\boxed{2}$, ~~$f^{-1}(V) = U$ is~~
$U = f^{-1}(V)$ is open.
$u \in U$
$\Rightarrow$ some $\delta > 0$ with
$B_\delta(u) \subset U$.

$f(B_\delta(u)) \subset f(U) = f(f^{-1}(V))$
$\subset V = B_\varepsilon(f(u))$

Proof (contd)

$A = \mathbb{R}^n$.

$\boxed{1} \Longrightarrow \boxed{2}$



Assume V closed. Let $U = f^{-1}(V)$.

Assume $\{u_k\} \subset U$ converge to $u \in \mathbb{R}^n$.

$f(u_k) \subset V$ converges to $f(u)$ and V closed,

so $f(u) \in V$. Hence, $u \in V = f^{-1}(\underset{\substack{\ni \\ u}}{U})$.

So U closed.