Math411 - 01/30/12

Topology of $\mathbb{R}^n$ (10.3)

1/

Topology = extra structure on space defining which sets in the space are "open"

Here: Only look at the "standard" topology on $\mathbb{R}^n$ induced by metric structure

Definition: $u \in \mathbb{R}^n$ $r > 0$

Open ball $B_r(u) = \{v \in \mathbb{R}^n \mid dist(u,v) < r\}$

Def: Complement $U \subseteq \mathbb{R}^n$ is

$\mathbb{R}^n \setminus U = \{u \in \mathbb{R}^n, u \notin U\}$ | Complement of Complement is U

~~Mo~~ The two most important complementary concepts

Open set $U \subseteq \mathbb{R}^n$ Closed set

Complementary:
U open $\Leftrightarrow$ $\mathbb{R}^n \setminus U$ closed
U closed $\Leftrightarrow$ $\mathbb{R}^n \setminus U$ open

Equivalent Defs:

$\boxed{1}$ U open $\Leftrightarrow$ for every pt $u \in U$ there is $r > 0$ such that $B_r(u) \subseteq U$

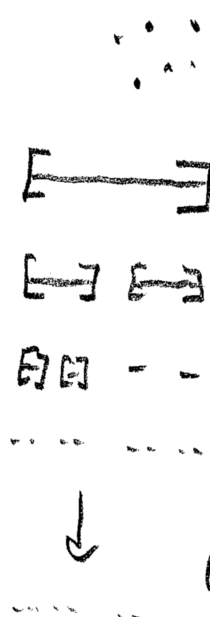
$\boxed{1B}$ $U = \operatorname{int} U$

easier to state for complement:

$\boxed{2}$ U closed $\Leftrightarrow$ for every sequence $\{u_k\} \subset U$ with limit $u \in \mathbb{R}$ has $u \in U$

<u>Examples</u>:

<u>Open</u>	<u>Closed</u>	<u>Remark</u>:
$\emptyset$ (empty set) $\quad \mathbb{R}^n$	$\emptyset \quad \mathbb{R}^n$	$\emptyset, \mathbb{R}^n$ are closed & open, such a set is called <u>clopen</u>. Clopen set is connected.



discrete/finite set of pts

closed interval

remove middle $1/3$

again

again

limit is Cantor set and closed

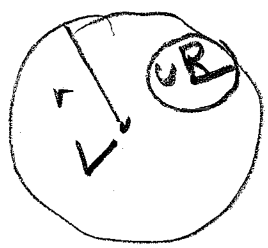
$\mathbb{Q}$ neither closed nor open ($\exists$ exists $u_k \to \sqrt{2}$, $u_k \in \mathbb{Q}$)

Most basic

Thm: Open ball $B_r(u)$ is open.

"open as ball" "open as set"

Proof: Let $v \in B_r(u)$. Let $R = r - \mathrm{dist}(u,v)$



Claim: $B_R(v) \subset B_r(u)$

Assume $w \in B_R(v)$.

$\mathrm{dist}(w,u) \leq \mathrm{dist}(w,v) + \mathrm{dist}\, v,u$

Triangle

$R + \mathrm{dist}(v,u)$

$\leq r$

so $w \in B_r(u)$. $\square$

Theorem: Def 1 and Def 2 are equivalent.

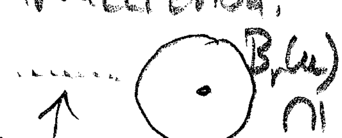
(Def 2 translated to complement defines when set is open) U open if ....

Proof:

" $\Longrightarrow$ "

Assume U open by Def 1. Assume there is a sequence $\{u_k\} \subseteq \mathbb{R}^n \setminus U$ with limit $u \in U$.

There is an $r > 0$ such that $B_r(u) \subset U$. All but finitely many $\{u_k\}$ are in $B_r(u)$. A contradiction. So $u \notin U$. So Def 2 of U open.



sequence in U's complement can't get into r-ball.

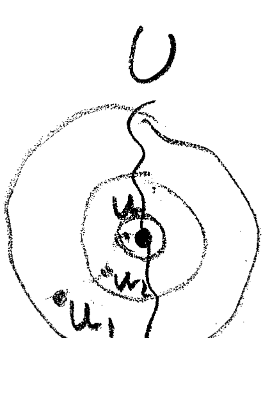
" $\Longleftarrow$ "

Assume Def 2 for U open holds, but U not open by Def 1.

Then there is a point $u \in U$ s.t. no ball $B_1(u), B_{1/2}(u), B_{1/4}(u), \ldots \subset U$.

There is a point $u_1, u_2, u_3, \ldots$ in each ball not in U. But $\text{dist}(u, u_1) < 1$, $\text{dist}(u, u_2) < \frac{1}{2}, \ldots$

so $u_k \to u \in U$, a contradiction.



Point $u \in \mathbb{R}^n$, take sequence of smaller balls $B_1(u), B_{1/2}(u), B_{1/4}(u) \ldots$.

If all balls ~~do~~ eventually are in U

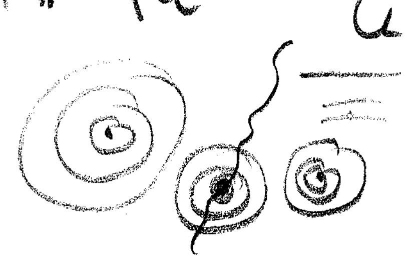
$\Rightarrow u$ interior point

If all balls eventually are in $\mathbb{R}^n \setminus U$

$\Rightarrow u$ exterior point

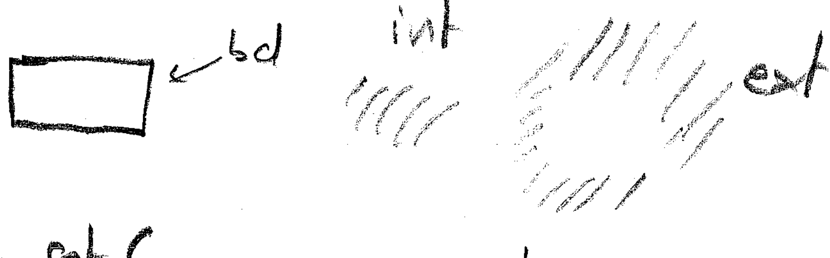
If undecided

$\Rightarrow u$ boundary.



If we use this as Def, then we get Def $\boxed{\mathbb{B}}$, of int

Examples: 1. Rectangle E = open intervals crossed $(a_1, b_1) \times \ldots \times (a_n, b_n)$



2. Cantor set C (closed) ....

ext = complement $\mathbb{R} \setminus C$

int = empty

boundary = Cantor set C

3. $\mathbb{Q}$ (dense)

int = empty

ext = empty

boundary = $\mathbb{R}$

Basic properties

Thm: 1. Finite intersection of open sets is open.
2. Union of open sets is open.

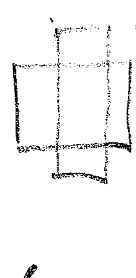
Proof:

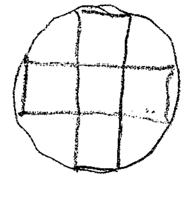
DeMorgan similar for closed
$$\bigcup_i U_i = \mathbb{R}^n \setminus \bigcap_i \mathbb{R}^n \setminus U_i$$

2. If $u \in \bigcup U_i$,
then there is i s.t. $u \in U_i$, hence there is $B_r(u) \subset U_i$,
and now $B_r(u) \subset \bigcup U_i$.

1. Left as exercise. Similarly, intersection of finitely many balls $B_{r_1}(u), \dots, B_{r_k}(u)$ is $B_r(u)$ where r is minimum of $r_1, \dots, r_k$. $\square$

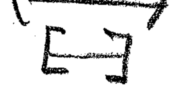
Example





union of all rectangles contained in ball is ball.

but:



intersection of all open intervals containing $[0,1]$ is closed.

Need finite for intersection.

<u>Different point of view:</u>

Def: Given space X, a set $\mathcal{O}$ of sets $U \subset X$ is called a topology if $\mathcal{O}$ is closed under

1. finite intersection
2. union
3. contains $\emptyset, X$.

A set in $\mathcal{O}$ is called open.

<u>Observation:</u> Partial order on topologies on X,

~~weak stro~~

$\mathcal{O} \subset \mathcal{O}'$

weaker stronger

<u>Topology induced by metric structure:</u>

weakest topology such that each ball (or alternatively, each open rectangle) is open.