To receive full credit, you must show all of your work. No books, notes or electronic equipment are allowed. You have 120 *minutes to complete the exam.*
Please write your name (last, first) and UID at the top right of each page.

Good luck!

35 P. 1. Answer each of the following questions with yes/no/can't tell. No justification necessary.

1. Let $\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field with div $\vec{G} \neq 0$. Is there a vector field $\vec{F}$ such that $\vec{G} = \text{curl } \vec{F}$?
   Solution: No. If there were such a vector field $\vec{F}$, then div $\vec{G} = \text{div}(\text{curl } \vec{F}) = 0$, a contradiction.
2. Let $\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field with div $\vec{G} \neq 0$. Is there a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ such that grad $f = \vec{G}$?
   Solution: Can't tell. For $f(x, y, z) = x^2$, we have $\text{div}(\text{grad} f) \neq 0$. However, $\vec{G}(x, y, z) = (x, x, 0)$ is a vector field with div $\vec{G} \neq 0$ and integrating $\vec{G}$ along different paths from $(0, 0, 0)$ to $(1, 0, 0)$ gives different values, so it cannot be the gradient of a scalar field.
3. Let $\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field with curl $\vec{G} = 0$. Is the function $f : \mathbb{R}^3 \to \mathbb{R}$ such that grad $f = \vec{G}$ uniquely determined?
   Solution: No, uniquely determined only up to constant, e.g., $f(x, y, z) = 0$ and $f(x, y, z) = 1$ have the same gradient grad $f = \vec{G} = 0$.
4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Riemann-integrable function with $\int f = 0$ such that $f(0) = 1$. Is f a non-negative function?
   Solution: Can't tell. $f(0) = 1$ and $f(x) = 0$ for $x \neq 0$ fulfills conditions and is non-negative. $f(0) = 1$, $f(1) = -1$ and $f(x) = 0$ for $x \neq 0, 1$ fulfills conditions and is not non-negative.
5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous and Riemann-integrable function with $\int f = 0$ such that $f(0) = 1$. Is f a non-negative function?
   Solution: No. By continuity, letting $\epsilon = 1/2$, there is a $\delta > 0$ such that $f(x) > 1/2$ on a δ-ball around 0. So in particular, $f(x) > 1/2$ on some non-zero volume rectangle around 0. If f were non-negative, then f is everywhere larger or equal to $\frac{1}{2}$ times the characteristic function on the non-zero volume rectangle, and hence $\int f$ is larger or equal to $\frac{1}{2}$ times the volume of the rectangle, so non-zero.
6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1/2^k$ for $x = m/2^k$ with $0 < m < 2^k$, m odd, $k \geq 0$ and $f(x) = 0$ otherwise. Is f Riemann-integrable?
   Solution: Yes. f has compact support (namely, [0,1]), is bounded (namely by 1) and is continuous everywhere except for the countable set of numbers 0, 1, and $m/2^k$. So f is Riemann-integrable by theorem.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ for $x = m/2^k$ with $0 < m < 2^k$, m odd, $k \geq 0$ and $f(x) = 0$ otherwise. Is f Riemann-integrable?

Solution: No. f is not continuous on (0,1], hence not Riemann-integrable by theorem. A more direct way to see it, is to notice that the infimum of f on any interval in P_k between $\frac{1}{2}$ and 1 will be zero, and the supremum at least $\frac{1}{2}$. So $U(f, P_k) - L(f, P_k) \geq \frac{1}{4}$ for all k.

30 P. 2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto ((y-1)e^x, (x-1)e^y)$.

1. Show that there is an open set $\mathcal{U}$ containing $(1, 1)$ such that f restricted to $\mathcal{U}$ is invertible with the inverse function g being continuously differentiable.
Solution: We have

$$Df(x, y) = \begin{pmatrix} (y-1)e^x & e^x \\ e^y & (x-1)e^y \end{pmatrix}, \quad Df(1, 1) = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix},$$

so $\det(Df(1, 1)) = -e^2 \neq 0$, and, hence f is locally invertible at $(1, 1)$. The value of $f(1, 1)$ is $(0, 0)$, so $g(0, 0) = (1, 1)$.

2. Compute

$$\lim_{t \to 0} \frac{g(2t, 3t) - (1, 1)}{t}.$$

Solution: Using $g(0, 0) = (1, 1)$ from above, this is the differential quotient for g at the point (0,0) with respect to the direction $(2, 3)$. We have

$$Dg(0, 0) = (Df(1, 1))^{-1} = \begin{pmatrix} 0 & 1/e \\ 1/e & 0 \end{pmatrix},$$

so $Dg(0, 0) \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = (3/e, 2/e)$ is the value of the limit.

30 P. 3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous Riemann-integrable function and let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Answer each of the following questions giving an argument if the answer is yes or counterexample otherwise.

1. Is $f \circ g$ defined by $(f \circ g)(x) = f(g(x))$ bounded?
Solution: Yes. If f is bounded by M, then so is $f \circ g$.

2. Does $f \circ g$ have compact support, i.e., is the closure of the set $(f \circ g)^{-1}(\mathbb{R} \setminus \{0\})$ compact, or equivalentely, is the set $(f \circ g)^{-1}(\mathbb{R} \setminus \{0\})$ bounded?
Solution: Example: $g(x) = x$. Counterexample: Define f by $f(x) = 1 - |x|$ for $|x| < 1$ and $f(x) = 0$ for $|x| \geq 1$. Let $g(x) = 0$, then $(g \circ f)(x) = 1$.

3. Is $f \circ g$ continuous?
Solution: Yes, composition of continuous functions is continuous.

4. Is $f \circ g$ Riemann-integrable?
Solution: Example and counterexample as in previous question on compact support.

25 P. 4. Let C be the closed curve $x + y - z = 0$, $x^2 + y^2 - (z/11)^2 - 1 = 0$ oriented clockwise when viewed from above. Let $\vec{F}$ be the vector field given by $\vec{F}(x, y, z) = (z, z, 0)$. Use Stoke's Theorem to compute $\int_C \vec{F} \cdot d\vec{r}$.

Hint: What is the normal vector of the plane $x + y - z = 0$?

Solution: We have curl $\vec{F} = (-1, 1, 0)$. Let S be the surface that is a subset of the plane $x + y - z = 0$ and bounded by C. Note that the normal vector is parallel to $(1, 1, -1)$ (or better $(-1, -1, 1)$ to induce the specified orientation on C), so the area element is $d\vec{A} = (1, 1, -1)\lambda \; du \; dv$ where λ is some scalar. By Stoke's Theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl } \vec{F} \cdot d\vec{A} = \int_S (-1, 1, 0) \cdot (1, 1, -1)\lambda \; du \; dv = \int_S 0 \; du \; dv = 0.$$

30 P. 5. Let $f_i : [0, 1] \to \mathbb{R}$ be a sequence of Riemann-integrable functions uniformly converging to $f : [0, 1] \to \mathbb{R}$. Prove that f is Riemann-integrable.

Hint: Show that, if $|f(x) - f_i(x)| < \epsilon/4$ for all x in $[0, 1]$, then $|U(f, P_k) - U(f_i, P_k)| < \epsilon/4$ and $|L(f, P_k) - L(f_i, P_k)| < \epsilon/4$. Pick k large enough such that $U(f_i, P_k) - L(f_i, P_k) < \epsilon/2$.

Remark: Here P_k denotes the standard partition of $[0, 1]$ by intervals $[t2^{-k}, (t + 1)2^{-k}]$ with t an integer. $U(f, P_k)$, respectively, $L(f, P_k)$ denotes the upper, respectively, lower Darboux sum with respect to P_k, i.e., the sum of the supremum, respectively, infimum of f on each interval in P_k multiplied with the length of the interval. We define a function $f : [0, 1] \to \mathbb{R}$ to be Riemann-integrable if the limit of $U(f, P_k)$ and $L(f, P_k)$ as $k \to \infty$ are the same.

Solution: Let $\epsilon > 0$. By uniform convergence, we can pick i large enough such that $|f(x) - f_i(x)| < \epsilon/4$. This means that the absolute value of the difference of the infimum, respectively, supremum of $f(I)$ and $f_i(I)$ is at most $(\epsilon/4) \cdot 2^{-k}$ for an interval in P_k. The number of intervals of the standard partition P_k of the interval $[0, 1]$ is 2^k, hence we have

$$|U(f, P_k) - U(f_i, P_k)| < \epsilon/4 \qquad \text{and} \qquad |L(f, P_k) - L(f_i, P_k)| < \epsilon/4. \tag{1}$$

Because f_i is Riemann-integrable, we can pick k large enough such that

$$|U(f_i, P_k) - L(f_i, P_k)| < \epsilon/2. \tag{2}$$

Using the triangle inequality on (1) and (2), we get

$$|U(f, P_k) - L(f, P_k)| < \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon.$$

Together with the fact that $U(f, P_k)$ is decreasing and $L(f, P_k)$ is increasing in k, this implies that f is Riemann-integrable.

Total: 150