Practice Problems for Final
Math 411, Section 201, Spring 2012

Here, Riemann-integrable is in the sense of the lecture notes meaning that the lower and upper Darboux sums converge to the same value with respect to the standard partition P_k of $\mathbb{R}^n$ by translations of the rectangle $[0, 2^{-k}]^n$ by vectors with entries being multiples of 2^{-k}. You can use the theorem characterizing Riemann-integrable functions as bounded functions $\mathbb{R}^n \to \mathbb{R}$ with compact support and with set of discontinuities having Lebesgue-measure zero.

1. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. What extra condition does g need to satisfy such that for every Riemann-integrable $f : \mathbb{R}^n \to \mathbb{R}$, $g \circ f$ is Riemann-integrable as well? Give a proof in both directions, i.e., prove that $g \circ f$ is Riemann-integrable if the condition is satisfied and give an f such that $g \circ f$ is not Riemann-integrable if the condition is not satisfied.

2. Show that $f(x) = \sin(1/x)$ for $x \in (0, 1)$, $f(x) = 0$ otherwise, is Riemann-integrable.

3. Prove that given Riemann-integrable functions $f_i, g_i : \mathbb{R}^n \to \mathbb{R}$

$$f_0 \leq f_1 \leq \ldots \leq f_n \leq \ldots \leq f \leq \ldots \leq g_n \leq \ldots \leq g_1 \leq g_0$$

   such that $\lim \int f_i = \lim \int g_i$, the function f is also Riemann-integrable.

4. Compute the volume of the torus obtained by revolving a circle of radius 2 in the $x - y$ plane with center having distance 5 from the x-axis around the x-axis.

5. Compute the volume of all points inside the hyperboloid $x^2 + y^2 - z^2 = 1$ and between the planes $z = +1$ and $z = -1$.

6. Compute the volume of a parallelepid spanned by the vectors $(1, 2, 3)$, $(4, 2, -1)$, and $(2, -2, 2)$.

7. Find a sequence of non-negative functions $f_i : \mathbb{R} \to \mathbb{R}$ with $\int f_i = 0$, such that $\sum f_i$ is not Riemann-integrable. Hint: Recall that the characteristic function of $\mathbb{Q} \cap [0, 1]$ did not have Jordan content/Riemann-measure 0.

8. Integrate $x^3 + y^3 + z^3$ over the upper hemiball $z > 0, x^2 + y^2 + z^2 < 1$.

9. Let f_i be a sequence of Riemann-integrable functions $\mathbb{R} \to \mathbb{R}$ uniformly converging to 0. Does $\int f_i$ converge, and if yes, to zero?

10. Let $f(x, y, z) = (2x, -y, x)$ be a vector field. Is there a vector field g such that $\text{curl} g = f$?

11. Let $f(x, y, z) = (0, 0, x)$ be a vector field. Find g such that $\text{curl} g = f$.