To receive full credit, you must show all of your work. No books, notes or electronic equipment are allowed. You have 120 *minutes to complete the exam.*
Please write your name (last, first) and UID at the top right of each page.

Good luck!

[35 P.] 1. Answer each of the following questions with yes/no/can't tell. No justification necessary.

1. Let $\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field with div $\vec{G} \neq 0$. Is there a vector field $\vec{F}$ such that $\vec{G} = \text{curl } \vec{F}$?
2. Let $\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field with div $\vec{G} \neq 0$. Is there a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ such that grad $f = \vec{G}$?
3. Let $\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field with curl $\vec{G} = 0$. Is the function $f : \mathbb{R}^3 \to \mathbb{R}$ such that grad $f = \vec{G}$ uniquely determined?
4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Riemann-integrable function with $\int f = 0$ such that $f(0) = 1$. Is f a non-negative function?
5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous and Riemann-integrable function with $\int f = 0$ such that $f(0) = 1$. Is f a non-negative function?
6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1/2^k$ for $x = m/2^k$ with $0 < m < 2^k$, m odd, $k \geq 0$ and $f(x) = 0$ otherwise. Is f Riemann-integrable?
7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ for $x = m/2^k$ with $0 < m < 2^k$, m odd, $k \geq 0$ and $f(x) = 0$ otherwise. Is f Riemann-integrable?

[30 P.] 2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto ((y-1)e^x, (x-1)e^y)$.

1. Show that there is an open set $\mathcal{U}$ containing $(1, 1)$ such that f restricted to $\mathcal{U}$ is invertible with the inverse function g being continuously differentiable.
2. Compute
$$\lim_{t \to 0} \frac{g(2t, 3t) - (1, 1)}{t}.$$

[30 P.] 3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous Riemann-integrable function and let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Answer each of the following questions giving an argument if the answer is yes or counterexample otherwise.

1. Is $f \circ g$ defined by $(f \circ g)(x) = f(g(x))$ bounded?
2. Does $f \circ g$ have compact support, i.e., is the closure of the set $(f \circ g)^{-1}(\mathbb{R} \setminus \{0\})$ compact, or equivalentely, is the set $(f \circ g)^{-1}(\mathbb{R} \setminus \{0\})$ bounded?
3. Is $f \circ g$ continuous?
4. Is $f \circ g$ Riemann-integrable?

25 P. 4. Let C be the closed curve $x + y - z = 0$, $x^2 + y^2 - (z/11)^2 - 1 = 0$ oriented clockwise when viewed from above. Let $\vec{F}$ be the vector field given by $\vec{F}(x, y, z) = (z, z, 0)$. Use Stoke's Theorem to compute $\int_C \vec{F} \cdot d\vec{r}$.
Hint: What is the normal vector of the plane $x + y - z = 0$?

30 P. 5. Let $f_i : [0, 1] \to \mathbb{R}$ be a sequence of Riemann-integrable functions uniformly converging to $f : [0, 1] \to \mathbb{R}$. Prove that f is Riemann-integrable.
Hint: Show that, if $|f(x) - f_i(x)| < \epsilon/4$ for all x in $[0, 1]$, then $|U(f, P_k) - U(f_i, P_k)| < \epsilon/4$ and $|L(f, P_k) - L(f_i, P_k)| < \epsilon/4$. Pick k large enough such that $U(f_i, P_k) - L(f_i, P_k) < \epsilon/2$.

Remark: Here P_k denotes the standard partition of $[0, 1]$ by intervals $[t2^{-k}, (t + 1)2^{-k}]$ with t an integer. $U(f, P_k)$, respectively, $L(f, P_k)$ denotes the upper, respectively, lower Darboux sum with respect to P_k, i.e., the sum of the supremum, respectively, infimum of f on each interval in P_k multiplied with the length of the interval. We define a function $f : [0, 1] \to \mathbb{R}$ to be Riemann-integrable if the limit of $U(f, P_k)$ and $L(f, P_k)$ as $k \to \infty$ are the same.

Total: 150